Physics 5413: Chaos, fractals, and nonlinear dynamics – Project 1

due date Thursday, Aug 29, 2024

In this project, you will explore the behavior of the logistic map $x_{n+1} = f_A(x_n) = Ax_n(1-x_n)$ using analytic and computational techniques. For the numerical parts, you are encouraged to write your own computer code. However, this is not supposed to be the main task of the project. Therefore, for those who do not want to program, I have provided all necessary computer code (albeit in a very spartan form) on the course web page. For plotting results you can use whatever program you are familiar with, e.g., MS Excel, Origin, or Gnuplot (which is free and available on the course web page). Gnuplot is the official supported plotting program for this course.

Bifurcations and chaos in the logistic map (100 points + 15 BONUS points for part 3)

1. **Feigenbaum numbers**
   Find the first 6 bifurcation points A_n of the logistic map. A_n is the parameter where the cycle of length 2^{n-1} becomes unstable, and is replaced by a stable cycle of length 2^n. (In class, we have shown that $A_1 = 3$).

   (a) Study the 2-cycle and its stability analytically. Note that a 2-cycle of the logistic map $f_A(x)$ is a fixed point of the map $g_A(x) = f_A[f_A(x)]$. Calculate the bifurcation point A_2.

   (b) Determine the bifurcation points A_1 to A_6 numerically. This is most easily done by just watching the trajectories over a narrow interval of A values. Note that you will need rather high accuracy (error smaller than 10^{-6} or so) for the following steps.
   HINT: In this analysis you need to discard the transient behavior, i.e., you need to wait for a number of iterations until the system reaches steady behavior.

   (c) Calculate the Feigenbaum numbers

   $$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$

   and determine an approximate value for the Feigenbaum number $\delta = \lim_{n\to\infty} \delta_n$. (This number is universal for all maps with a quadratic maximum.) Estimate A_∞, i.e., the onset of chaos in this bifurcation sequence.

2. **Divergence of nearby trajectories**

   (a) For several parameter values $A = 0.5, 2.5, 3.2, 4$, determine how nearby trajectories diverge. To do so, numerically follow two trajectories x_n and y_n which start from nearby points with $|y_0 - x_0| \ll 1$. Monitor the behavior of $\Delta_n = |y_n - x_n|$. Does it increase or decrease with time? Can you guess the functional form of the n-dependence?
   HINT: Before the numerical analysis, think about how to choose the initial separation of the trajectories and the time interval to follow the trajectories. If you choose Δ_0 too large or follow the trajectory for too long you get an incorrect result. Why?

   (b) Plot $\Delta_n = |y_n - x_n|$ as a function of n. Fit the n-dependence of Δ_n to an exponential $\Delta_n = \Delta_0 \exp(\lambda n)$. The exponent λ is called a Lyapunov exponent. What value do you obtain for λ?
   HINT: Think about what the most appropriate type of plot and choice of axes is!

3. **Closed form solution of the logistic map for** $A = 4$ (15 BONUS POINTS)

Surprisingly, for $A = 4$, there is a closed form solution of the logistic map

$$x_n = \sin^2(\pi 2^n z_0) \quad \text{where} \quad z_0 = \frac{1}{\pi} \arcsin \sqrt{x_0}$$

(a) Show, that the above is indeed a solution of the logistic map for $A = 4$.

(b) Can you understand why nearby trajectories diverge so rapidly? Determine the Lyapunov exponent for $A = 4$ analytically and compare to what you found in part 2.