

Physics 5413: Chaos and Dynamics – Project 7

due Nov 21, 2024

A strange repeller for the tent map (100 pts + 5 bonus)

The tent map is defined on the closed interval $[0,1]$ via $x_{n+1} = f(x_n)$, where

$$f(x) = \begin{cases} rx & (x \leq 1/2) \\ r(1-x) & (x \geq 1/2) \end{cases}.$$

Here, we assume that the control parameter $r > 2$. Then some points get mapped outside the interval $[0,1]$. If we start with $x_0 \in [0,1]$ and $f(x_0) > 1$, then we say that x_0 has escaped after one iteration. Similarly, x_0 has escaped after n iterations, if $f^{(k)}(x_0) \in [0,1]$ for all $k < n$ but $f^{(n)}(x_0) > 1$.

- a) Find the set of initial conditions x_0 that escape after one iteration. Determine those that escape after two, three, and four iterations. Find the pattern.
- b) Describe the set of x_0 that never escape, the so-called invariant set. Illustrate this set by making a qualitative plot. The invariant set is called a strange repeller because it has fractal structure and it repels all points not in the set.
- c) Calculate the box dimension of the invariant set. Does it depend on r ? How?
- d) Show that the local Lyapunov exponent is positive at each point of the invariant set.
- e) Consider the case $r = 2$. Show that the flat distribution $P(x) = 1$ (for $0 \leq x \leq 1$) of points x is invariant under the tent map. (This means that $P(y) = 1$ with $y = f(x)$).
- f) How does this change for $r > 2$? Find the density of (surviving) points after two, three, four, ... iterations. Find the pattern and generalize to infinite iterations.

BONUS: Multifractal properties?

- g) Is the invariant set a multifractal? Evaluate the generalized dimensions D_q . (You can assume the point density to be constant, see above.)