Physics 6311: Statistical Mechanics - Final Exam

May 15, 2019

Problem 1: Two-level systems (50 pts)

a) Consider a single two-level system with states at energies 0 and ϵ. Use the canonical ensemble to calculate its specific heat as function of temperature. (30 pts)

b) A piece of material contains a large number N of such two-level systems. Their energies ϵ are randomly distributed between 0 and ϵ_{max} with a probability density $P(\epsilon) = C\epsilon^{\lambda-1}$ characterized by a positive exponent λ (C is the normalization constant). Determine the average specific heat of the entire sample for temperatures $k_B T \ll \epsilon_{max}$. (You do not need to evaluate constants given by dimensionless integrals). (20 pts)

Problem 2: Fermi gas with quartic energy-momentum relation (50 pts)

Consider a gas of N noninteracting spin-1/2 fermions at zero temperature in a cubic box of linear size L. The single-particle energies of these fermions are given by $\epsilon(\mathbf{k}) = A|\mathbf{k}|^4$ where $\mathbf{k}$ is the wave vector and A is a constant.

a) Find the Fermi energy ϵ_F as function of the density N/L^3. (25 pts)

b) Determine the total internal energy as a function of N and L. (25 pts)

Problem 3: Bose-Einstein condensation with absorption sites. (80 pts)

An ideal Bose gas of N nonrelativistic spin-0 particles of mass m is in a cubic box of linear size L. In addition, there are $N_A \ll N$ absorption sites on the surfaces of the box. Each absorption site can either be empty, or contain a single of the Bose particles. An absorbed particle has energy Δ. The absorbed particles are in equilibrium with the particles in the gas.

a) Starting from the Bose distribution, find the Bose-Einstein condensation temperature when no absorption sites are present, i.e., for $N_A = 0$. (You do not need to evaluate constants given by dimensionless integrals). (35 pts)

b) What is the value of the chemical potential μ of the Bose gas in the condensed phase? (5 pts)

c) Using the grand canonical ensemble, calculate the average number of absorbed particles as a function of temperature T and energy Δ, at this chemical potential μ. (20 pts)

d) Describe how does the critical temperature for Bose-Einstein condensation change due to the absorption sites? Find its dependence on N_A and Δ. (Hint: The total particle number is the sum of the number of particles in the gas and the number of absorbed particles.) (10 pts)

e) Discuss the limits $\Delta \to -\infty$ and $\Delta \to \infty$. (10 pts)

Problem 4: Mean-field theory of Blume-Capel model (120 pts)

Each site of a square lattice is occupied by a spin 1, i.e., by a variable S_i that can take values $-1, 0, +1$. The Hamiltonian reads

$$H = -J \sum_{\langle ij \rangle} S_i S_j + \Delta \sum_i S_i^2 - \mu_B B \sum_i S_i$$

where the first sum runs over all pairs of nearest neighbors and $J > 0$. The so-called crystal field energy Δ can take positive or negative values. This Hamiltonian is called the Blume-Capel model.

a) Analyze the system at zero temperature and zero magnetic field: What is the ground state for negative Δ? What is the ground state for large positive $\Delta \gg J$? Compute the ground state energies. (20 pts)

b) The system undergoes a phase transition as a function of Δ at fixed J, zero temperature, and zero magnetic field. At what value of Δ does the transition happen? Is the transition continuous or of first order? (10 pts)

c) Derive a mean-field approximation of the Hamiltonian. (20 pts)

d) Solve the mean-field Hamiltonian and derive the mean-field equation. (25 pts)

e) Solve this mean-field equation for $B = 0$ and find the critical temperature T_c as function of J and Δ. (You do not need to solve the final transcendental equation for T_c.) (20 pts)

f) Discuss how T_c behaves in the limits $\Delta \to \infty$ and $\Delta \to -\infty$. (15 pts)

g) Describe how you would decide whether the transition is continuous or of first order. You do not actually have to perform the calculation. (10 pts)