Physics 6311: Statistical Mechanics - Test Preparation Homework 13

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due date: Tuesday, Nov 28, 2023

Problem 1: Random window panes (10 points)

A machine in a factory making glass window panes is malfunctioning. As a result, it is producing rectangular windows of random size. Specifically, the horizontal and vertical sizes of the window are independent random quantities. They can take values between 0 and 3 m with a constant probability density.

a) What are minimum and maximum possible areas?
b) Write down the probability densities for the horizontal and vertical sides of the window
c) Calculate the average area $\langle A \rangle$ of the produced windows and its standard deviation.
d) Derive the probability density of A. (Hint: Be careful with the integration bounds when transforming and integrating over the δ-function)
e) What is the most likely area?

Problem 2: Ideal gas with movable piston (10 points)

A classical ideal gas of N particles is in a cylindrical vessel of cross section A. The top of the vessel is closed by a movable piston of mass M.

a) Calculate the partition function for the system consisting of gas + piston in the canonical ensemble.

b) Determine the equation of state, the average volume and the heat capacity.

c) Discuss which heat capacity you are actually calculating.

Problem 3: Debye phonons in two dimension (10 points)

Consider a thin film (two-dimensional solid) of N atoms and linear size L. This solid has 3N phonon modes. Within the Debye model, the phonon frequencies are $\omega_{\vec{k}} = c|k|$ for $0 \leq \omega_{\vec{k}} < \Omega_D$. Here, c is the speed of sound.

a) Calculate the density of states $g(\epsilon)$.

b) Determine the Debye frequency Ω_D.

c) Calculate the internal energy and the specific heat for low temperatures ($k_B T \ll \hbar\Omega_D$) and for high temperatures ($k_B T \gg \hbar\Omega_D$).

Problem 4: Mean-field theory of an Ising antiferromagnet (10 points)

Consider an Ising model ($S_i = \pm 1$) given by a Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i S_j - \mu_B B \sum_i S_i$$

with a negative exchange interaction J on a cubic lattice.

a) What is the ground state in the absence of a field ($B = 0$)? What is the ground state for large field B? Where does it change?

b) Derive a mean-field theory for this model by introducing two average spin values for the two sublattices you have identified in part a)

c) Find the critical temperature for the onset of antiferromagnetism in the absence of a field (the so-called Neel temperature).

d) Determine the total magnetic susceptibility $\chi = \partial m / \partial B$ above the Neel temperature. What is the value of the Weiss temperature? Discuss its sign.