Physics 6311: Statistical Mechanics - Homework 4

due date: Tuesday, Sep 19, 2023

Problem 1: Maxima of entropy (10 points)

Consider the entropy of a discrete probability distribution given in terms of the probabilities p_i ($i = 1...N$). Determine which p_i lead to the maximum entropy under the following constraints (Hint: Use Lagrange multipliers to enforce the constraints.):

a) Normalization $\sum_i p_i = 1$

b) Normalization $\sum_i p_i = 1$ and fixed average $\langle a \rangle = \sum_i p_i a_i$ of a quantity A with values a_i.

Problem 2: Shannon entropy of independent random variables (10 points)

Consider two discrete, jointly distributed random variables X and Y with values x_i and y_j, respectively. The joint probability of X having the value x_i and Y having the value y_j is p_{ij}.

a) Show that if X and Y are statistically independent, then the Shannon entropy S_s of the joint distribution is the sum of the Shannon entropies of the reduced distributions of X and Y

b) Generalize the derivation to the case on M jointly distributed variables $X^{(m)}$ with $m = 1 \ldots M$.

Problem 5: Shannon entropy of N spin-1 atoms (5 points)

Consider a lattice with $N \gg 1$ spin-1 atoms. Each atom can be in one of the three spin states $S_z = -1, 0, +1$ with equal probability. The states of different atoms are independent of each other. Calculate the Shannon entropy of this system.

Problem 4: Atoms on a lattice (15 points)

Consider a lattice having N regular lattice sites as well as N interstitial lattice sites. The lattice is occupied by N identical atoms. An atom on a regular site has energy 0 while an atom on an interstitial site has energy ϵ. Use the microcanonical ensemble to analyze this system.

a) Determine the number Ω of microstates as a function of the number N_i of atoms on interstitial sites.

b) Relate N_i to the energy E and compute the temperature T as a function of N and E.

c) Express E as a function of T and N, and find the specific heat.