Physics 6311: Statistical Mechanics - Homework 5

due date: Tuesday, Sep 30, 2025

Problem 1: Microcanonical ideal gas (20 points)

Consider a gas of nonrelativistic, non-interacting, distinguishable quantum particles in a cubic box of linear size L (with periodic boundary conditions). The energy-momentum relation of a single particle is $\epsilon = \vec{p}^{\,2}/(2m)$.

a) Determine the (single-particle) wave functions. What are the allowed $\vec{k}$ values? What is the volume per state in $\vec{k}$-space?

b) Calculate the number of microstates as a function of the total (system) energy E.
(Hint: First calculate the number of states with energies less than E and then take the derivative with respect to E.)

c) Calculate the entropy as function of the energy.

d) Calculate the temperature and the caloric equation of state (energy-temperature relation).

e) Calculate the thermodynamic equation of state (relation between p, V, T).

Problem 2: Spin-$\frac{1}{2}$ in a magnetic field (10 points)

Consider N (distinguishable) $S = \frac{1}{2}$ quantum spins in a magnetic field $\mathbf{B} = B\mathbf{e}_z$ in z-direction. The Hamiltonian is given by

$$\hat{H} = -Bg\mu_B \sum_{i=1}^{N} S_i^{(z)}, \qquad S_i^{(z)} \pm \frac{1}{2}.$$

Here g is the gyromagnetic factor and μ_B is the Bohr magneton. The spins are coupled to a heat bath at temperature T.

a) Use the canonical ensemble to calculate the partition function, Helmholtz free energy, the entropy, the internal energy and the specific heat as functions of temperature.

b) Calculate the magnetization $M = g\mu_B \sum_{i=1}^{N} \langle S_i^{(z)} \rangle$ and the magnetic susceptibility $\chi = (\partial M/\partial B)_T$ as functions of T and B. Determine the zero-field susceptibility $\lim_{B\to 0} \chi(B, T)$.

Problem 3: Model of DNA (10 points)

A simple model of the DNA double helix molecule is analogous to a zipper: a chain of N links each of which can be open or closed. A closed link has energy ϵ_0, and an open link has energy $\epsilon_1 > \epsilon_0$. Replication of the DNA starts with the opening of the "zipper". Assume that it can only open from one end (say the left), i.e., a link can only be open if all links left of it are also open.

a) Calculate the partition function for this DNA model.

b) Find the average number of open links n as a function of N and temperature T .

c) Discuss the behavior of N in the limits of high and low temperatures.