Physics 6311: Test preparation homework 7

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due date: Friday, Oct 10, 2023

Problem 1: Quantum mechanical three-level system (12 points)

A quantum-mechanical system has three energy eigenstates $|0\rangle, |-1\rangle$ and $|1\rangle$ with energies $\epsilon_0 = 0$, $\epsilon_1 = \epsilon_{-1} = \epsilon$ where $\epsilon < 0$.

a) Use the canonical ensemble to determine the average energy and the heat capacity as functions of temperature.

b) Compute the entropy and find its behavior in the limits of low and high temperatures.

c) Calculate the occupation probabilities p_0, p_1 and p_{-1} of the three levels as functions of temperature.

d) At what temperature is $p_0/p_1 = 2$? Discuss the sign of this temperature.

Problem 2: Broadening of spectral lines (16 points)

The atoms of a star emit light. The emission frequency of a particular element is ν_0 if the atom is a rest. Due to the thermal motion the observed frequency is shifted (Doppler effect) to

$$\nu = \nu_0 \left(1 - \frac{v}{c}\cos\theta\right)$$

where v is the velocity of the atom and θ is the angle between the directions of motion and observation. Assume the atoms in the star's atmosphere have mass m and can be approximately described as a classical ideal gas at temperature T.

a) Compute the mean observed frequency by directly averaging ν over the appropriate velocity distribution.

b) Compute the variance of the observed frequency.

c) Derive an expression for the intensity distribution $\rho(\nu)$ of the spectral line, i.e., the probability density of the observed frequency. Compare with the results of parts a) and b). [Hint: This problem can be solved either in spherical coordinates or in cartesian coordinates. The latter approach may be a bit easier.)

Problem 3: Ideal gas in linear potential well (12 points)

Consider a classical ideal gas of N non-interacting particles at temperature T. The particles are subject to the potential energy $U(\vec{r}) = A|\vec{r}|$ (A is a constant).

a) Calculate the partition function and the Helmholtz free energy of the gas.

b) Determine the internal energy and the specific heat. Compare with the equipartition theorem.

c) Calculate the particle density $n(\vec{r})$ of the gas (particles per volume) as a function of the position. [Hint: the particle density $n(\vec{r})$ can be related to a reduced probability density of the single-particle phase space density $\rho(\vec{r}, \vec{p})$.]