Physics 413: Midsemester test 8 Mar 2010

Problem 1: Sum and difference of two Gaussian distributed random variables (50 points)

The random variables X and Y are independent and Gaussian distributed with zero average and standard deviations of 1. Here, we study the statistics of the random variables $V = X + Y$ and $W = X - Y$.

a) Calculate the averages $\langle v \rangle$, $\langle w \rangle$, and $\langle v\, w \rangle$. (15 pts)
b) Find the joint probability density $P_{VW}(v, w)$ for the random variables V and W. (Hint: It may be useful to introduce a characteristic function $f_{VW}(k_1, k_2)$.) (30 pts)
c) Are V and W independent? (5 pts)

Problem 2: Spin-1 paramagnet (100 points)

A paramagnetic material contains N localized (and thus distinguishable), non-interacting spins with quantum numbers $S = 1$ and $S^{(z)} = -1, 0, +1$. In an external magnetic field $\vec{B}$, the Hamiltonian reads

$$H = -\sum_{i=1}^{N} \mu \vec{B} \cdot \vec{S}_i / \hbar$$

where the constant μ is the magnetic moment associated with the spin.

a) Calculate the canonical partition function and the Helmholtz free energy. (30 pts)
b) Determine the magnetization (magnitude and direction), $\langle \vec{M} \rangle = \sum_i \mu \langle \vec{S}_i \rangle / \hbar$, as a function of temperature and discuss its behavior in the limits $T \to 0$ and $T \to \infty$. (30 pts)
c) Compute the magnetic susceptibility $\chi = (\partial M / \partial B)_T$. (20 pts)
d) Find the leading high-temperature behavior of χ and compute the Curie constant. (20 pts)

Problem 3: Ideal gas in the gravitational field (100 points)

Consider a non-relativistic classical ideal gas of N indistinguishable particles at temperature T in a cylindrical vessel of cross section A and height H. The gas particles are in a gravitational potential $E_{pot} = mgz$ where m is the mass of a particle, g is the free fall acceleration and z is the vertical coordinate. (Assume H to be large, $mgH \gg k_B T$)

a) Find the canonical partition function and Helmholtz free energy. (30 pts)

b) Find the contribution of the potential energy term to the total energy and the specific heat. (30pts)

c) Calculate the particle density $n(z)$ as a function of z. (Hint: $n(z)$ is a reduced probability density of the single-particle phase space density $\rho(\vec{r}, \vec{p})$.) (30 pts)

d) What is the approximate ratio between the air density at sea level and at an altitude of 10000m? Use the following approximate values: $g \approx 10\text{ms}^{-2}$, $m \approx 4 * 10^{-26}$kg (mass of a nitrogen molecule), $k_B \approx 4/3 * 10^{-23}$J/K, $T = 300$K. (10pts)

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standard form of a normalized Gaussian distribution: $P(x) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2}(x-x_0)^2/\sigma^2}$
$\int_0^\infty dx\, e^{-ax} = 1/a,\quad \int_0^\infty dx\, x e^{-ax} = 1/a^2 \quad \int_0^\infty dx\, x^2 e^{-ax} = 2/a^3$
$\int_{-\infty}^\infty dx\, e^{-ax^2+bx} = \sqrt{\pi/a}\, e^{b^2/(4a)},\quad \int_{-\infty}^\infty dx\, x^2\, e^{-ax^2} = \sqrt{\pi/(4a^3)}$
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots,\quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2} + \ldots,\quad \sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{6} + \ldots$