Physics 6311: Mid-semester test — Oct 12, 2023

Problem 1: Spin-1/2 in a magnetic field (60 points)

Consider a single $S = 1/2$ quantum spin in a magnetic field $\mathbf{B} = B\hat{\mathbf{k}}$ and coupled to a heat bath at temperature T. The Hamiltonian is given by

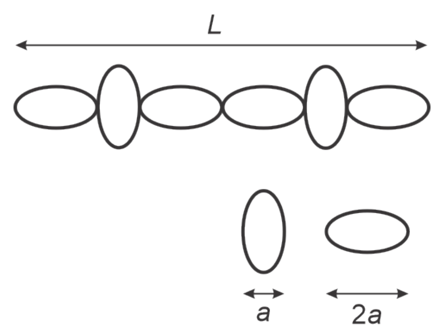
$$H = -Bg\mu_B S_z, \qquad S_z = \pm\frac{1}{2}.$$

Here g is the gyromagnetic factor and μ_B is the Bohr magneton.

a) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy.
b) Calculate the magnetization $m = g\mu_B\langle S_z\rangle$ as a function of T and B.
c) Find the values of m for $T = 0$ and $T \to \infty$ at fixed nonzero B.
d) Calculate the magnetic susceptibility $\chi = (\partial m/\partial B)_T$
e) Find the behavior (not just the limit) of χ at high temperatures.

Problem 2: One-dimensional polymer (70 points)

A one-dimensional polymer is formed by connecting N ellipsoid-shaped molecules into a one-dimensional chain. Each molecule has two ways of connecting to the polymer (as shown in the figure). It can align either its long axis (length $2a$) or its short axis (length a) with the direction of the polymer chain. A molecule connected along the long axis has energy ϵ_0, a molecule connected along the short axis has energy $\epsilon_0 + \epsilon$ with $\epsilon > 0$.



a) Using the canonical ensemble, find the partition function of this polymer as a function of temperature T.
b) Find the probabilities p_l and p_s that a given molecule is connected along the long axis and the short axis, respectively.
c) Calculate the average length L of the polymer as a function of temperature and number of molecules.
d) What is value of the average length at $T = 0$ and $T \to \infty$?
e) At what temperature is the average length $L = 5aN/4$. What does the sign of T mean?

Problem 3: Nonharmonic classical ideal gas (70 points)

Consider a gas of N non-interacting, indistinguishable, classical particles in a cubic box of linear size L. The particles have non-harmonic energy-momentum relation, i.e., the Hamiltonian reads

$$H = \sum_{i=1}^{N} A|\vec{p}_i|^s \qquad (A, s > 0)$$

a) Calculate the canonical partition function and the Helmholtz free energy of this gas. [Hint: Work in spherical coordinates.]

b) Calculate the caloric equation of state (energy-temperature relation) and the specific heat C_V at constant volume. Compare to the equipartition theorem.

c) Calculate the pressure p and find the thermodynamic equation of state (relation between p, V, T).

d) Compute the specific heat C_p at constant pressure.

$$\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x) = 1/\cosh^2(x)$$

$$\int_0^\infty dx\, x^2 e^{-Ax^s} = \frac{1}{s} A^{-3/s}\,\Gamma(3/s) \qquad (\Gamma \text{ denotes the Gamma function.})$$