1a)

$$\left.\begin{aligned} \frac{\partial}{\partial y}(x^2-2y) &= -2 \\ \frac{\partial}{\partial x}(-2x) &= -2 \end{aligned}\right\} \quad du_a \text{ exact}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}(y^2) &= 2y \\ \frac{\partial}{\partial x}(-2x) &= -2 \end{aligned}\right\} \quad du_b \text{ not exact}$$

b)

$$u_a = \frac{1}{3}x^3 - 2xy + \text{const}$$

c) $\underline{u_a}$:

$$\left.\begin{aligned} \int_{0,0}^{2,0} du_a &= \int_0^2 dx\,(x^2-2y)\Big|_{y=0} = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3} \\ \int_{(2,0)}^{(2,2)} du_a &= \int_0^2 dy\,(-2x)\Big|_{x=2} = -8 \end{aligned}\right\} -\frac{16}{3}$$

$$\left.\begin{aligned} \int_{(0,0)}^{(0,2)} du_a &= \int_0^2 dy\,(-2x)\Big|_{x=0} = 0 \\ \int_{(0,2)}^{(2,2)} du_a &= \int_0^2 dx\,(x^2-2y)\Big|_{y=2} = \frac{8}{3} - 8 = -\frac{16}{3} \end{aligned}\right\} -\frac{16}{3}$$

$\Rightarrow$ both path give same result

$\underline{u_b}$:

$$\left.\begin{aligned} \int_{(0,0)}^{(2,0)} du_b &= \int_0^2 dx\; y^2 \Big|_{y=0} = 0 \\ \int_{(2,0)}^{(2,2)} du_b &= \int_0^2 dy\; (-2x)\Big|_{x=2} = -8 \end{aligned}\right\} -8$$

$$\left.\begin{aligned} \int_{(0,0)}^{(0,2)} du_b &= \int_0^2 dy\; (-2x)\Big|_{x=0} = 0 \\ \int_{(0,2)}^{(2,2)} du_b &= \int_0^2 dx\; y^2\Big|_{y=2} = 8 \end{aligned}\right\} +8$$

the two path give different results

2 a) $I_a = \int_0^\infty dx \; x \; \delta(e^{-x} - 2)$

$e^{-x_0} = 2 \quad \Rightarrow \quad x_0 = -\ln 2 < 0$

outside of integration interval

$I_a = 0$

2 b) $I_b = \int_{-\infty}^{\infty} dx \; \cos(\pi x) \; \delta(1 - x^2)$

$x_0^2 = 1 \quad \Rightarrow \quad x_0 = \pm 1$

use $\delta(f(x)) = \sum_{\text{zeros}} \delta(x - x_0) / |f'(x_0)|$

here $f(x) = 1 - x^2 \qquad f'(x) = -2x$

$I_b = \cos(\pi) / |-2| + \cos(-\pi) / |2| = -1$

3.) Derive general result for Gaussian integral

$$\underline{I} = \int d^d x \quad e^{-\vec{x}^T \underline{M} \vec{x} - 2\vec{b}\cdot\vec{x}}$$

with symmetric matrix $\underline{M}$ and d-vector $\vec{b}$

substitute $\quad \vec{x} = \vec{y} - \underline{M}^{-1}\vec{b}$

$$+\vec{x}^T \underline{M} \vec{x} + 2\vec{b}\cdot\vec{x} = \vec{y}^T \underline{M} \vec{y} - \cancel{\vec{b}^T \underline{M}^{-1} \underline{M} \vec{y}}$$

$$- \cancel{\vec{y}^T \underline{M} \underline{M}^{-1} \vec{b}} + \vec{b}^T \underline{M}^{-1} \underline{M} \underline{M}^{-1} \vec{b}$$

$$+ \cancel{2\vec{b}\cdot\vec{y}} - 2\vec{b}^T \underline{M}^{-1} \vec{b}$$

$$= \vec{y}^T \underline{M} \vec{y} - \vec{b}^T \underline{M}^{-1} \vec{b}$$

$$\underline{I} = \int d^d y \quad e^{-\vec{y}^T \underline{M} \vec{y} + \vec{b}^T \underline{M}^{-1} \vec{b}}$$

$$= e^{\vec{b}^T \underline{M}^{-1} \vec{b}} \prod_{\nu=1}^{d} \sqrt{\frac{\pi}{\lambda_\nu}} \quad \leftarrow \text{ eigenvalues of } \underline{M}$$

Here : $\underline{M} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -A/2 \\ -B/2 \end{pmatrix}$

<u>eigenvectors</u> $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_1 = \frac{1}{2}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda_2 = \frac{3}{2}$

$$\underline{M}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

check

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \checkmark$$

$$\underline{I} = e^{\frac{1}{12}\begin{pmatrix} A \\ B \end{pmatrix}^T \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}} \frac{\pi}{\sqrt{\frac{1}{2}\,\frac{3}{2}}}$$

$$\underline{I} = e^{\frac{1}{3}\left(A^2 + B^2 + AB\right)} \frac{2}{\sqrt{3}}\pi$$

Problem 4:

a) not in equilibrium, not in steady state

b) equilibrium

c) nonequilibrium steady state (almost)

d) not in equilibrium, not in steady state
(for large-scale properties approximately in nonequilibrium steady state)

e) equilibrium

f) nonequilibrium steady state