1a) $\frac{\partial}{\partial y}(x^2-y^2) = -2y$
$\frac{\partial}{\partial x}(-2x) = -2$
$\Big\}$ du_a not exact

$\frac{\partial}{\partial y}(y^2) = 2y$
$\frac{\partial}{\partial x}(2xy) = 2y$
$\Big\}$ du_b exact

b) $u_b = y^2 x + \text{const}$

c)

$\underline{du_a}$: $0,0 \to 2,0 \to 2,2$

$$\Delta u_a = \int_{x=0}^{2} (x^2-y^2)\,dx + \int_{y=0}^{2} (-2x)\,dy = \frac{8}{3} - 8 = -\frac{16}{3}$$

$0,0 \to 0,2 \to 2,2$

$$\Delta u_a = \int_0^2 dy(-2x) + \int_0^2 (x^2-y^2)\,dx = \frac{8}{3} - 8 = -\frac{16}{3}$$

$\underline{du_b}$: $0,0 \to 2,0 \to 2,2$

$$\Delta u_b = \int_{x=0}^{2} y^2\,dx + \int_{y=0}^{2} 2xy\,dy = \emptyset + 8 = 8$$

$0,0 \to 0,2 \to 2,2$

$$\Delta u_b = \int_{y=0}^{2} 2xy\,dy + \int_{x=0}^{2} y^2\,dx = \emptyset + 8 = 8$$

$u_b(2,2) - u_b(0,0) = 8$

2 a) $$I_a = \int_0^\infty dx \; x \; \delta(e^{-x} - 2)$$

$e^{-x_0} = 2 \qquad x_0 = -\ln(2) < 0$

outside integration interval $\Rightarrow I_a = 0$

b) $$I_b = \int_{-\infty}^{\infty} dx \cos(\pi x) \; \delta(x^2 - 4)$$

$x_0^2 = 4 \Rightarrow x_0 = \pm 2$

use $$\delta[f(x)] = \sum_{\text{zeros}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

here $f(x) = x^2 - 4 \qquad f'(x) = 2x$

$$I_b = \frac{\cos(2\pi)}{|4|} + \frac{\cos(-2\pi)}{|-4|} = \frac{1}{2}$$

3.) Derive general result for Gaussian integral

$$\underline{I} = \int d^d x \; e^{-\vec{x}^T \underline{M} \vec{x} - 2 \vec{b} \cdot \vec{x}}$$

with symmetric matrix $\underline{M}$ and d-vector $\vec{b}$

substitute $\vec{x} = \vec{y} - \underline{M}^{-1} \vec{b}$

$$+\vec{x}^T \underline{M} \vec{x} + 2\vec{b} \cdot \vec{x} = \vec{y}^T \underline{M} \vec{y} - \cancel{\vec{b}^T \underline{M}^{-1} \underline{M} \vec{y}}$$
$$- \cancel{\vec{y}^T \underline{M}\, \underline{M}^{-1} \vec{b}} + \vec{b}^T \underline{M}^{-1} \underline{M}\, \underline{M}^{-1} \vec{b}$$
$$+ \cancel{2 \vec{b} \cdot \vec{y}} - 2 \vec{b}^T \underline{M}^{-1} \vec{b}$$
$$= \vec{y}^T \underline{M} \vec{y} - \vec{b}^T \underline{M}^{-1} \vec{b}$$

$$\underline{I} = \int d^d y \; e^{-\vec{y}^T \underline{M} \vec{y} + \vec{b}^T \underline{M}^{-1} \vec{b}}$$

$$= e^{\vec{b}^T \underline{M}^{-1} \vec{b}} \prod_{\nu=1}^{d} \sqrt{\frac{\pi}{\lambda_\nu}} \quad \leftarrow \text{eigenvalues of } \underline{M}$$

Here : $\underline{M} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -A/2 \\ -B/2 \end{pmatrix}$

<u>eigenvectors</u> $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_1 = \frac{1}{2}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda_2 = \frac{3}{2}$

$$\underline{M}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Check

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \checkmark$$

$$\underline{I} = e^{\frac{1}{12}\begin{pmatrix} A \\ B \end{pmatrix}^T \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}} \quad \frac{\pi}{\sqrt{\frac{1}{2}\,\frac{3}{2}}}$$

$$\underline{I} = e^{\frac{1}{3}(A^2 + B^2 + AB)} \quad \frac{2}{\sqrt{3}}\pi$$

Problem 4:

a) not in equilibrium, not in steady state

b) equilibrium

c) nonequilibrium steady state (almost)

d) not in equilibrium, not in steady state
(for large-scale properties approximately in nonequilibrium steady state)

e) equilibrium

f) nonequilibrium steady state