Problem 10.1

①

Generalized Bose gas

a) $$\langle N \rangle = (2S+1) \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_{\vec{k}} - \mu)} - 1} \qquad \varepsilon_{\vec{k}} = A |\hbar \vec{k}|^z$$

$$= (2S+1) \left(\frac{L}{2\pi}\right)^d S_{d-1} \int dk\, k^{d-1} \frac{1}{e^{\beta(\varepsilon_{\vec{k}} - \mu)} - 1}$$

S_{d-1} $\hat{=}$ surface of d-dimensional unit sphere

$$k = \frac{1}{\hbar} (\varepsilon / A)^{\frac{1}{z}} \qquad dk = \frac{\varepsilon^{\frac{1}{z} - 1}}{\hbar z A^{1/z}} d\varepsilon$$

$$\langle N \rangle = (2S+1) \left(\frac{L}{2\pi}\right)^d S_{d-1} \frac{1}{z} \left(\frac{1}{\hbar A^{1/z}}\right)^d \int d\varepsilon\, \varepsilon^{d/z - 1} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

$$g(\varepsilon) = (2S+1) \frac{L^d}{(2\pi)^d} S_{d-1} \frac{1}{z \hbar^d A^{d/z}} \varepsilon^{d/z - 1} = VB\, \varepsilon^{d/z - 1}$$

b) Set $\mu = 0$

$$\langle N_{max} \rangle = VB \int_0^\infty d\varepsilon\, \varepsilon^{d/z - 1} \frac{1}{e^{\beta\varepsilon} - 1} \qquad \begin{aligned} x &= \beta \varepsilon \\ d\varepsilon &= k_B T\, dx \end{aligned}$$

$$= VB (k_B T)^{d/z} \underbrace{\int_0^\infty dx \frac{x^{d/z - 1}}{e^x - 1}}_{I}$$

BE condensation if integral converges

upper bound: integrand $\sim e^{-x}$ $\Rightarrow$ converges

lower bound: $\int_0 dx\, x^{d/z - 2}$ converges if

$$d/z - 2 > -1 \quad \Rightarrow \quad \boxed{d > z}$$

BE condensation if $d > z$

c) critical temperature when $N = \langle N_{max} \rangle$

$$N = VB\,(k_B T)^{d/z} \cdot I$$

$$k_B T_c = \left(\frac{N}{VBI}\right)^{z/d}$$

d) Below T_c : $\mu = 0$, condensate does not contribute to $\langle E \rangle$, $\langle p \rangle$.

$$\langle E \rangle = \int d\varepsilon\, g(\varepsilon)\, \frac{\varepsilon}{e^{\beta\varepsilon} - 1} = VB\,(k_B T)^{d/z+1} \int_0^\infty dx\, \frac{x^{d/z}}{e^x - 1}$$

$$= VB\,(k_B T)^{d/z+1} \cdot I_2$$

$$C_V \sim \frac{\partial \langle E \rangle}{\partial T} = \left(\frac{d}{z} + 1\right) \frac{\langle E \rangle}{T} \sim T^{d/z}$$

e) $$S = \int_0^T dT' \, \frac{C_V'}{T'} = \frac{z}{d}\left(\frac{d}{z} + 1\right) \frac{\langle E \rangle}{T} = \left(1 + \frac{z}{d}\right) \frac{\langle E \rangle}{T}$$

$$A = \langle E \rangle - TS = -\frac{z}{d} \langle E \rangle$$

$$p = \left(-\frac{\partial A}{\partial V}\right)_T = \frac{z}{d} \langle E \rangle / V \sim T^{d/z+1}$$

Problem 10.2

$$u(\varepsilon) = \frac{1}{\pi^2 \hbar^3 c^3} \quad \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1}$$

maximum

$$\frac{d}{d\varepsilon} \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1} = \frac{3\varepsilon^2}{e^{\beta\varepsilon} - 1} - \frac{\beta\varepsilon^3 e^{\beta\varepsilon}}{(e^{\beta\varepsilon} - 1)^2}$$

$$= \frac{3\varepsilon^2(e^{\beta\varepsilon} - 1) - \beta\varepsilon^3 e^{\beta\varepsilon}}{(e^{\beta\varepsilon} - 1)^2} \stackrel{!}{=} 0$$

$$3 e^{\beta\varepsilon} - 3 = \beta\varepsilon\, e^{\beta\varepsilon}$$

Solve $3(e^x - 1) = x e^x \quad \Rightarrow \quad x = 2.8214$

$$\varepsilon = 2.8214 \, k_B T = 1.17 \cdot 10^{-22} \text{ Ws}$$

$$\varepsilon = \hbar\omega = 2\pi\hbar\nu = \frac{2\pi\hbar c}{\lambda} \qquad \omega = 1.1 \cdot 10^{12} \text{ s}^{-1}$$

$$\lambda = \frac{2\pi\hbar c}{\varepsilon} = 1.7 \text{ mm}$$

Problem 10.3

Radiation of Betelgeuse

a) energy density $u(\varepsilon)$

$$u(\varepsilon) = \frac{1}{\pi^2 c^3 \hbar^3} \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1}$$

maximum at $\frac{du}{d\varepsilon} = 0$

$$\frac{d}{d\varepsilon} \frac{\varepsilon^3}{e^{\beta\varepsilon}-1} = 0 \quad \Rightarrow \quad \frac{d}{dx} \frac{x^3}{e^x-1} = 0$$

$$\frac{3x^2}{e^x-1} - \frac{x^3 e^x}{(e^x-1)^2} = 0 \Rightarrow 3x^2(e^x-1) = x^3 e^x$$

$(3-x)e^x = 3$ numerical solution $x \approx 2.82$

$\beta\varepsilon_{max} = 2.82$ $\quad T_{Bet} = \frac{\varepsilon_{max}}{2.82\, k_B} \approx 3290\ K$

b) $P_{Bet} \approx 10^4 P_{sun} = 10^4 \times 3.85 \times 10^{26}\, W = 3.85 \times 10^{30}\, W$

Stefan-Boltzmann:

$$P_{Bet} = 4\pi r_{Bet}^2 \times \frac{\pi^2}{60} \frac{(k_B T)^4}{\hbar^3 c^2}$$

$$r_{Bet} = \sqrt{\frac{15}{\pi^3} \frac{\hbar^3 c^2}{(k_B T)^4} P_{Bet}} = 2.15 \times 10^{11}\ m$$

c) maximum of spectrum is in the red color range

radius much larger than sun