# Physics 6311: Statistical Mechanics - Homework Solutions 11

## 11.1 Thermodynamics of magnons

a) (degeneracy: $f$)

$$N = f \sum_{\vec{k}} \frac{1}{e^{\beta \varepsilon_k} - 1}$$

$$= f \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta \varepsilon_k} - 1} = f \frac{V}{2\pi^2} \int dk\, k^2 \frac{1}{e^{\beta \varepsilon_k} - 1}$$

$$\varepsilon = \hbar D k^2 \qquad k = \sqrt{\frac{1}{\hbar D} \varepsilon} \qquad dk = \frac{1}{2} \frac{1}{\sqrt{\hbar D}} \frac{1}{\sqrt{\varepsilon}}$$

$$N = f \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \frac{1}{2} \int d\varepsilon \sqrt{\varepsilon} \frac{1}{e^{\beta \varepsilon} - 1}$$

$$g(\varepsilon) = \frac{f}{2} \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \sqrt{\varepsilon}$$

b)

$$E = \int_0^\infty d\varepsilon\, g(\varepsilon)\, \varepsilon \frac{1}{e^{\beta \varepsilon} - 1}$$

$$= \frac{f}{2} \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \int d\varepsilon\, \varepsilon^{3/2} \frac{1}{e^{\beta \varepsilon} - 1}$$

$$= \frac{f}{2} \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} (k_B T)^{5/2} \underbrace{\int dx\, x^{3/2} \frac{1}{e^x - 1}}_{\text{const}}$$

$$E \sim (k_B T)^{5/2}$$

$$C_v = \tfrac{5}{2} E/T \sim (k_B T)^{3/2}$$

**11.2: Phonons in liquid $^4$Helium**

a)

$$\Theta_D = \frac{\hbar\omega_D}{k_B} = \frac{\hbar}{k_B}\left(\frac{6\pi^2 c^3 N}{V}\right)^{1/3} = \frac{\hbar}{k_B}\left(\frac{6\pi^2 c^3 \rho}{m}\right)^{1/3} = 19.8\text{K}$$

with $m = 4 * 1.67*10^{-27}$kg (mass of Helium atom).

b) no transversal phonons, degeneracy factor is 1

$$\frac{C_V}{N} = \frac{4}{5}\pi^4 k_B \left(\frac{T}{\Theta_D}\right)^3$$

$$\frac{C_V}{mN} = \frac{4}{5}\pi^4 \frac{k_B}{m} \left(\frac{T}{\Theta_D}\right)^3$$

$$\frac{C_V}{mN} = 0.021\ (T/K)^3\ J/(gK)$$

**11.3: Phonons in 1D chain**

a)

$$H = \sum_{i=j}^{N} \frac{p_j^2}{2m} + \frac{A}{2} \sum_{j=1}^{N} (x_j - x_{j+1})^2 \ .$$

discrete Fourier transformation: $x_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} \tilde{x}_k$ with $k = -\pi, -\pi + \frac{2\pi}{L}, ..., \pi$ and analogously for $p_j$

$$\begin{aligned} H &= \sum_{j=1}^{N} \frac{p_j^2}{2m} + \frac{A}{2} \sum_{j=1}^{N} \left( x_j^2 + x_{j+1}^2 - 2x_j x_{j+1} \right) \\ &= \sum_k \frac{\tilde{p}_k^2}{2m} + \frac{A}{2} \sum_k \left( 2|\tilde{x}_k|^2 - 2e^{-ik} |\tilde{x}_k|^2 \right) \\ &= \sum_k \frac{\tilde{p}_k^2}{2m} + \frac{A}{2} \sum_k (2 - 2\cos k) |\tilde{x}_k|^2 \end{aligned}$$

Comparison with harmonic oscillator: $\omega_k = (\frac{A}{m}(2 - 2\cos k))^{1/2}$, for small k: $\omega_k = (\frac{A}{m})^{1/2}|k|$

b)

transform $k$-sum into integral,

$$U = \frac{L}{2\pi} \int_{-\pi/2}^{\pi/2} dk \ \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1}$$

substitute $x = \beta\hbar\omega_k$, for low $T$, the $x$-integral can be extended to infinity.

$$\begin{aligned} U &= \frac{L}{2\pi\beta} \frac{2}{\beta\hbar(A/m)^{1/2}} \int_0^\infty dx \ \frac{x}{e^x - 1} \\ &= \text{const} * T^2 \\ C_V &= 2 * \text{const} * T \end{aligned}$$