5.1 a) $\psi(\vec{k}) = \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{r}}$ $\quad k_x, k_y, k_z = \frac{2\pi}{L} n$ ← integer

volume per state in N-particle $\vec{k}$-space

$$\Delta V_k = \left(\frac{2\pi}{L}\right)^{3N}$$

b) $$E(\vec{k}) = \frac{\hbar^2}{2m}\left(k_1^2 + \ldots + k_{3N}^2\right)$$

surface of const E = hypersphere of radius

$\sqrt{\frac{2mE}{\hbar^2}} = k_{max}$, volume of sphere $V_s = C\, k_{max}^{3N}$

$$\tilde{\Omega} = \frac{V_s}{\Delta V_k} = \left(\frac{L}{2\pi}\right)^{3N} C \sqrt{\frac{2mE}{\hbar^2}}^{\,3N}$$

$$\Omega \sim \frac{\partial \tilde{\Omega}}{\partial E} \sim L^{3N} E^{\frac{3N}{2}-1}$$

c) $$S = k_B \ln \Omega = k_B \ln\left(\text{const} \times L^{3N} E^{\frac{3N}{2}-1}\right)$$

d) $$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V = k_B\left(\frac{3N}{2}-1\right)\frac{1}{E} . \Rightarrow E = \frac{3}{2} N k_B T$$

(keeping only terms $\sim N$)

e) $$\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_E = k_B N \frac{1}{V} \Rightarrow pV = N k_B T$$

(use $V = L^3$)

5.2. a) $H = T + V = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell\cos\theta$

$$= \frac{p^2}{2m\ell^2} - mg\ell\cos\theta$$

$$Q = \int \frac{d\theta\, dp}{h}\, e^{-\frac{\beta}{2}\frac{p^2}{m\ell^2}}\, e^{\beta mg\ell\cos\theta}$$

$$= Q_p\, Q_\theta$$

b) $\langle \dot{\theta} \rangle = 0$ by symmetry

$$\langle \dot{\theta}^2 \rangle = \langle p^2 \rangle / m^2\ell^4$$

$$\langle p^2 \rangle = \frac{1}{Q_p}\int dp\, e^{-\beta p^2/2m\ell^2}\, p^2 = m\ell^2 k_B T$$

$$\langle \dot{\theta}^2 \rangle = \frac{k_B T}{m\ell^2} = \sigma_{\dot{\theta}}^2$$

c) $Q_p = \sqrt{2\pi m\ell^2 k_B T} = \sqrt{2\pi m\ell^2/\beta}$

(neglect $V_0 = -mg\ell$) $Q_\theta \approx \int d\theta\, e^{-\frac{1}{2}\beta mg\ell\theta^2} = \sqrt{2\pi k_B T/mg\ell} = \sqrt{2\pi/\beta mg\ell}$

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\ln Q = -\frac{\partial}{\partial\beta}\left(\ln Q_p + \ln Q_\theta\right) = \frac{1}{2}k_B T + \frac{1}{2}k_B T$$

$$\langle E \rangle = k_B T \qquad C = \left(\frac{\partial E}{\partial T}\right) = k_B$$

d) $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 \mp \cdots$

$$Q_\theta = \int d\theta\, e^{-\frac{1}{2}\beta mg\ell\theta^2}\, e^{\frac{1}{24}\beta mg\ell\theta^4} =$$

$$= \int d\theta\, e^{-\frac{1}{2}\beta mg\ell\theta^2}\left(1 + \frac{1}{24}\beta mg\ell\theta^4\right)$$

$$= \sqrt{2\pi/\beta mg\ell} + \frac{1}{24}\, 3\sqrt{2\pi/(\beta mg\ell)^3}$$

$$= \sqrt{2\pi/\beta mg\ell}\left(1 + \frac{1}{8}\,\frac{1}{\beta mg\ell}\right)$$

5.2d) continued

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\left(\ln Q_p + \ln Q_\theta\right)$$

expand ln

$$= \frac{k_B T}{2} + \left(-\frac{\partial}{\partial\beta}\right)\left[\ln\sqrt{2\pi/\beta mg\ell} + \ln\left(1 + \frac{1}{8}\frac{1}{\beta mg\ell}\right)\right]$$

$$= \frac{k_B T}{2} + \frac{k_B T}{2} - \frac{\partial}{\partial\beta}\left(\frac{1}{8}\frac{1}{\beta mg\ell}\right)$$

$$= k_B T + \frac{1}{8}\frac{1}{\beta^2 mg\ell} = k_B T + \frac{1}{8}\frac{(k_B T)^2}{mg\ell}$$

$$C = k_B + \frac{k_B T^2}{4mg\ell}$$