7.1a) $Q = 1 + 2e^{-\beta\varepsilon}$

$$E = -\frac{\partial}{\partial\beta}\ln Q = \frac{2\varepsilon e^{-\beta\varepsilon}}{1+2e^{-\beta\varepsilon}} = \frac{2\varepsilon}{e^{\beta\varepsilon}+2}$$

$$C = \left(\frac{\partial E}{\partial T}\right) = \frac{2\varepsilon^2 e^{\beta\varepsilon}}{(e^{\beta\varepsilon}+2)^2}\frac{1}{k_B T^2} = 2k_B\left(\frac{\varepsilon}{k_B T}\right)^2 \frac{e^{\beta\varepsilon}}{(e^{\beta\varepsilon}+2)^2}$$

b) $$S = \frac{1}{T}(E - A) = \frac{1}{T}\left(\frac{2\varepsilon}{e^{\beta\varepsilon}+2} + k_B T \ln(1+2e^{-\beta\varepsilon})\right)$$

- $T \to 0$, $\beta \to \infty$ $\Rightarrow$ $e^{\beta\varepsilon} \to 0$, $e^{-\beta\varepsilon} \to \infty$ $(\varepsilon < 0)$

$$S = \frac{1}{T}\left(\cancel{\varepsilon} + k_B T \ln 2 - \cancel{k_B T \beta\varepsilon}\right) = k_B \ln 2$$ (ground state doubly degenerate)

- $T \to \infty$, $\beta \to 0$

$$S = \frac{1}{T}\left(\underbrace{\cancel{\tfrac{2}{3}\varepsilon}}_{\to 0} + k_B T \ln 3\right) = k_B \ln 3$$ (all states equally likely)

c) $$p_0 = \frac{1}{1+2e^{-\beta\varepsilon}} \qquad p_1 = p_{-1} = \frac{e^{-\beta\varepsilon}}{1+2e^{-\beta\varepsilon}}$$

d) $$p_0/p_1 = 1/e^{-\beta\varepsilon} = e^{\beta\varepsilon} = 2 \qquad \beta\varepsilon = \ln 2$$

$$k_B T = \frac{\varepsilon}{\ln 2} < 0$$ population inversion!

7.2 - single-particle phase-space density

$$\rho(p,q) = \frac{1}{Q_1} e^{-\beta p^2/2m}$$

- transform to velocity, integrate out $q$

$$P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\beta m \vec{v}^2/2}$$

a) $$\langle \nu \rangle = \nu_0 - \nu_0/c \underbrace{\langle v_x \rangle}_{0} = \nu_0 \qquad (v_x = v\cos\theta)$$

b) $$\langle (\nu-\nu_0)^2 \rangle = \frac{\nu_0^2}{c^2} \langle v_x^2 \rangle = \frac{\nu_0^2}{c^2} \frac{k_B T}{m}$$

c) $$P_\nu(\nu) = \int_{-\infty}^{\infty} dv_x \, P(v_x) \, \delta\left(\nu - \nu_0 + \frac{\nu_0}{c} v_x\right)$$

$$= \int_{-\infty}^{\infty} dv_x \left(\frac{m}{2\pi k_B T}\right)^{1/2} e^{-\frac{m}{2}\frac{v_x^2}{k_B T}} \delta\left(v_x - \frac{c}{\nu_0}(\nu_0 - \nu)\right) \frac{c}{\nu_0}$$

$$P_\nu(\nu) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \frac{c}{\nu_0} e^{-\frac{m}{2k_B T}\left(\frac{c^2}{\nu_0^2}(\nu-\nu_0)^2\right)}$$

7.3 a)

$$Q_N = \frac{1}{N!} Q_1^N \qquad Q_1 = \int \frac{d^3p\, d^3r}{h^3} e^{-\beta\left(\frac{p^2}{2m} + A|\vec{r}|\right)}$$

$$Q_1 = Q_{1kin} Q_{1pot} = \frac{1}{h^3}(2\pi m k_B T)^{3/2} \, 4\pi \int_0^\infty dr\, r^2 e^{-\beta A r}$$

$$= \frac{4\pi}{h^3}(2\pi m k_B T)^{3/2} \, 2\left(\frac{k_B T}{A}\right)^3$$

$$A = -k_B T \ln Q_N = -k_B T N \ln\left(\frac{8\pi}{h^3}(2\pi m k_B T)^{3/2}\left(\frac{k_B T}{A}\right)^3\right) + k_B T \ln$$

b)

$$E = -\frac{\partial}{\partial\beta}\ln Q = N\left(\frac{3}{2}k_B T + 3k_B T\right) = \frac{9}{2} N k_B T$$

$$C_V = \frac{9}{2} N k_B$$

(each kinetic d.o.f gives $\frac{1}{2}k_B$
each potential ~ $k_B$)

c)

$$n(\vec{r}) = N\int d^3p\, \rho_1(\vec{p}, \vec{r}) = \frac{N}{8\pi}\left(\frac{A}{k_B T}\right)^3 e^{-\beta A|\vec{r}|}$$