8.1 a) $$Q_G = \left(1 + e^{-\beta(\varepsilon_r - \mu)}\right)^N = \left(1 + e^{\beta\mu}\right)^N$$

$$N_r = \frac{\partial}{(\beta\mu)} \ln Q_G = \frac{N e^{\beta\mu}}{1 + e^{\beta\mu}} = \frac{N}{e^{-\beta\mu} + 1}$$

b) $$Q_G = \left(1 + e^{-\beta(\varepsilon - \mu)}\right)^N$$

$$N_i = \frac{\partial}{\partial(\beta\mu)} \ln Q_G = \frac{N e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}} = \frac{N}{e^{\beta(\varepsilon-\mu)} + 1}$$

c) $$N = N_i + N_r \qquad (\mu_i = \mu_r)$$

$$1 = \frac{1}{e^{-\beta\mu} + 1} + \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$(e^{-\beta\mu} + 1)(e^{\beta(\varepsilon-\mu)} + 1) = e^{\beta(\varepsilon-\mu)} + e^{-\beta\mu} + 2$$

$$e^{\beta(\varepsilon - 2\mu)} = 1$$

$$\mu = \varepsilon/2$$

d) in grand canonical ensemble

$$\langle E - \mu N \rangle = -\frac{\partial}{\partial\beta} \ln Q_G$$

$$\langle E_r - \mu N_r \rangle = -\frac{\partial}{\partial\beta} \ln\left(1 + e^{\beta\mu}\right)^N = \frac{-N\mu e^{\beta\mu}}{1 + e^{\beta\mu}}$$

$$\langle E_i - \mu N_i \rangle = \frac{N(\varepsilon - \mu) e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}$$

insert $\mu = \varepsilon/2$

$$\langle E_r - \tfrac{\varepsilon}{2} N_r \rangle = \frac{-N\varepsilon/2 \, e^{\beta\varepsilon/2}}{1 + e^{\beta\varepsilon/2}}$$

$$\langle E_i - \tfrac{\varepsilon}{2} N_i \rangle = \frac{N \varepsilon/2 \, e^{-\beta \varepsilon/2}}{1 + e^{-\beta \varepsilon/2}} = \frac{N \varepsilon/2}{1 + e^{\beta \varepsilon/2}}$$

$$\langle E \rangle = \langle E \rangle_r + \langle E_i \rangle = N \left[\varepsilon/2 - \frac{\varepsilon/2 \, e^{\beta \varepsilon/2}}{1 + e^{\beta \varepsilon/2}} + \frac{\varepsilon/2 \, e^{-\beta \varepsilon/2}}{1 + e^{-\beta \varepsilon/2}} \right]$$

$$= N\varepsilon \left[\frac{\frac{1}{2} + \cancel{\frac{1}{2} e^{\beta \varepsilon}} - \frac{1}{2} \cancel{e^{\beta \varepsilon/2}} + \frac{1}{2}}{1 + e^{\beta \varepsilon/2}} \right].$$

$$\boxed{\langle E \rangle = \frac{N\varepsilon}{1 + e^{\beta \varepsilon/2}}} \quad \text{as in HW4}$$