9.1 a) $S=0$ bosons

$n_a = 2 \quad n_b = 0 \qquad E = 2\varepsilon$

$n_a = 1 \quad n_b = 1 \qquad E = 0$

$n_a = 0 \quad n_b = 2 \qquad E = -2\varepsilon$

partition function

$$Q = e^{-\beta 2\varepsilon} + 1 + e^{\beta 2\varepsilon}$$

$$Q = 1 + 2\cosh(2\beta\varepsilon)$$

$$A = -k_B T \ln Q = -k_B T \ln\left(1 + 2\cosh(2\beta\varepsilon)\right)$$

$$\langle E \rangle = -\frac{\partial \ln Q}{\partial \beta} = \frac{-4\varepsilon \sinh(2\beta\varepsilon)}{1 + 2\cosh(2\beta\varepsilon)}$$

$$S = \frac{1}{T}(E - A) = k_B\left[-\frac{4\beta\varepsilon \sinh(2\beta\varepsilon)}{1+2\cosh(2\beta\varepsilon)} + \ln\left(1 + 2\cosh(2\beta\varepsilon)\right)\right]$$

$S = \frac{1}{2}$ fermions both in spin $\uparrow$ state

Pauli: only $n_a = 1$, $n_b = 1$ state allowed

$Q = 1 \qquad A = 0$

$E = 0 \qquad S = 0$

b) $S=0$ bosons

$n_a = 2 \quad n_b = 0 \quad E = 2\varepsilon$

$n_a = 1 \quad n_b = 1 \quad E = U$

$n_a = 0 \quad n_b = 2 \quad E = -2\varepsilon$

Canonical probabilities

$$P_1 = \frac{e^{-\beta 2\varepsilon}}{e^{\beta 2\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta U}}$$

$$P_2 = \frac{e^{-\beta U}}{e^{\beta 2\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta U}}$$

$$P_3 = \frac{e^{\beta 2\varepsilon}}{e^{\beta 2\varepsilon} + e^{-\beta 2\varepsilon} + e^{-\beta U}}$$

positive U suppresses state $(n_a = 1, n_b = 1)$
negative U prefers state $(n_a = 1, n_b = 1)$

- for $U \to -\infty$ this state becomes ground state

- for $U \to \infty$ state goes to $E \to \infty$ and drops out of thermodynamics

9.2 Quantum corrections to classical ideal gas

The classical (Boltzmann) limit corresponds to

$$\langle n\rangle = \frac{1}{e^{\beta(\epsilon-\mu)}+\delta} \ll 1$$

Expanding the denominator gives

$$\langle n\rangle = e^{-\beta(\epsilon-\mu)}(1-\delta e^{-\beta(\epsilon-\mu)}) + O[(e^{-\beta(\epsilon-\mu)})^3]$$

Average particle number and internal energy read

$$\begin{aligned} N &= \int d\epsilon\ a(\epsilon)\ e^{-\beta(\epsilon-\mu)}(1-\delta e^{-\beta(\epsilon-\mu)}) \\ U &= \int d\epsilon\ \epsilon\ a(\epsilon)\ e^{-\beta(\epsilon-\mu)}(1-\delta e^{-\beta(\epsilon-\mu)}) \end{aligned}$$

with

$$a(\epsilon) = \frac{V}{2\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2}\sqrt{2\epsilon}$$

The energy per particle is

$$\begin{aligned} \frac{U}{N} &= \frac{\int d\epsilon\ \epsilon^{3/2}\ e^{-\beta\epsilon}(1-\delta e^{-\beta(\epsilon-\mu)})}{\int d\epsilon\ \epsilon^{1/2}\ e^{-\beta\epsilon}(1-\delta e^{-\beta(\epsilon-\mu)})} \\ &= k_BT\frac{(3/4)\sqrt{\pi}\ (1-\delta e^{\beta\mu}/2^{5/2})}{(1/2)\sqrt{\pi}\ (1-\delta e^{\beta\mu}/2^{3/2})} \\ &= \frac{3}{2}k_BT\ (1+\delta e^{\beta\mu}\sqrt{2}/8) \end{aligned}$$

The factor $e^{\beta\mu}$ in the correction term can be calculated in lowest order, i.e. for the classical ideal gas.

$$e^{\beta\mu} = \frac{N}{V}\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} = \frac{N}{V}\lambda^3$$

where λ is the thermal wave length. Therefore

$$\frac{U}{N} = \frac{3}{2}k_BT\ (1+\delta N\lambda^3\sqrt{2}/(8V))$$