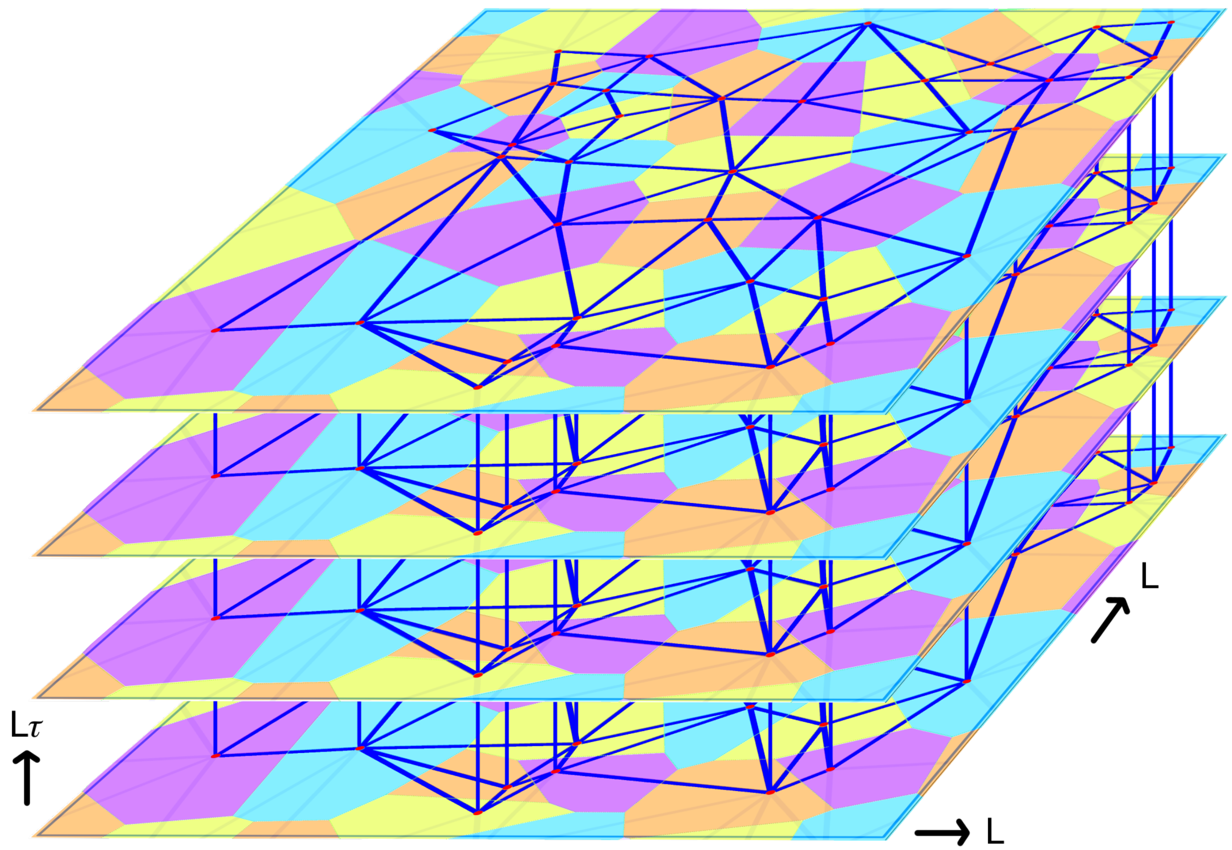
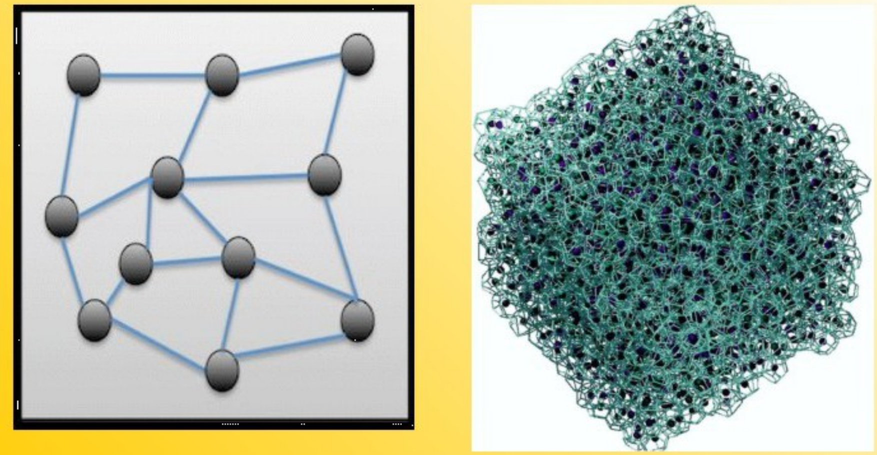
Critical Behavior and Collective Modes at the Superfluid Transition in Amorphous Systems

Thomas Vojta
Department of Physics, Missouri University of Science and Technology

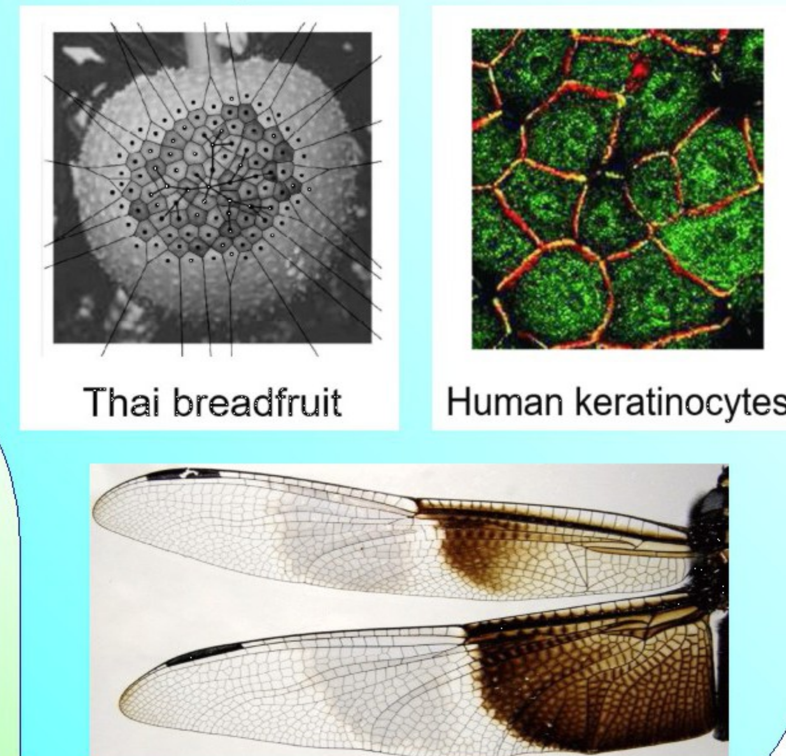


FQMT 24, Prague, July 22, 2024

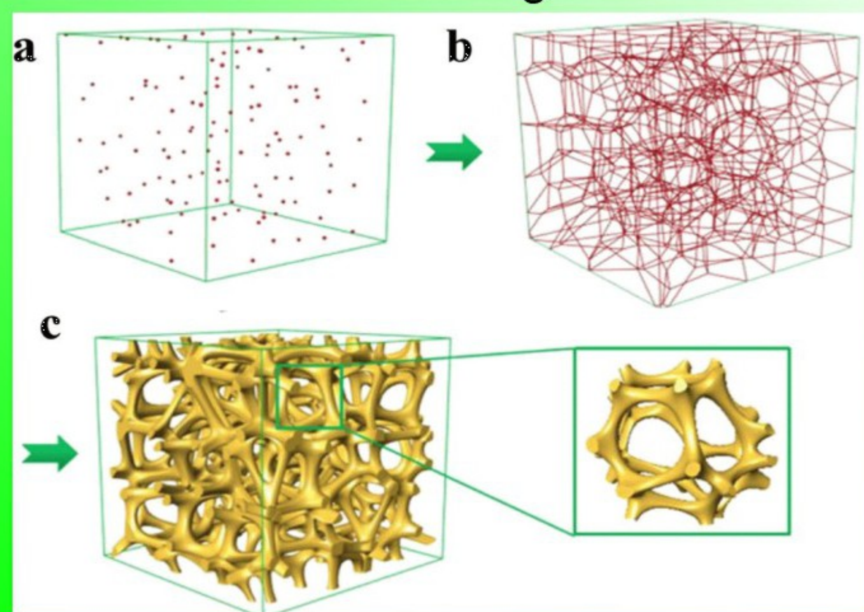
Amorphous solids and liquids



Biological cell structures



Additive manufacturing: bone scaffold



Outline

- Random lattices and hyperuniform disorder
- Superfluid-insulator transition on a random Voronoi lattice
- Amplitude (Higgs) mode puzzle

DMR-1828489
OAC-1919789
PHY-2309135

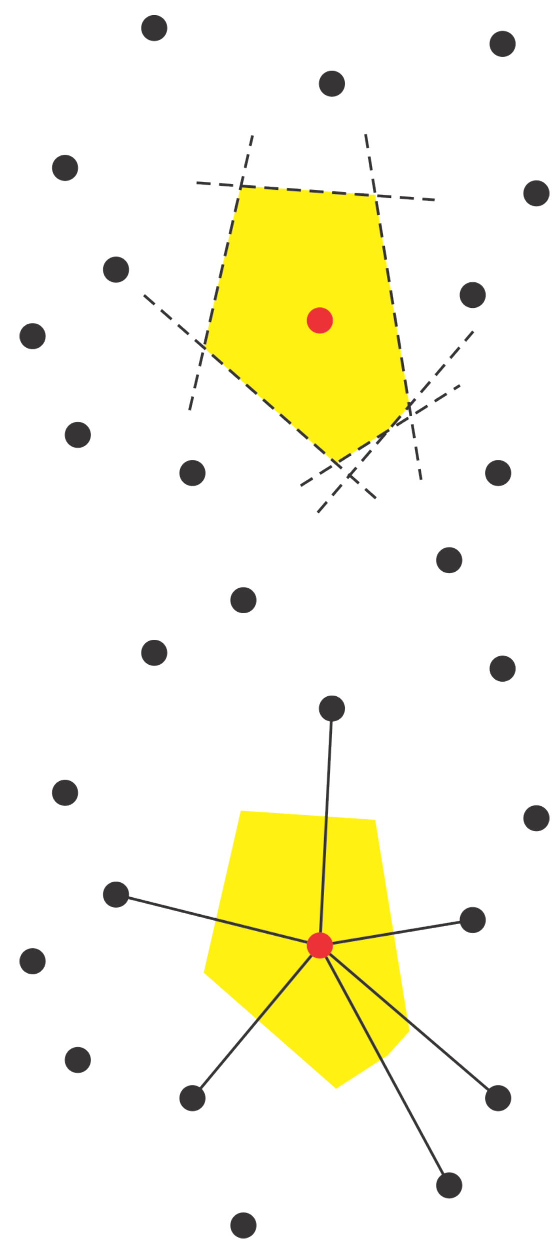
Vishnu PK

Martin Puschmann

Rajesh Narayanan

Random Voronoi-Delaunay lattice

- construct cell structure from set of random lattice sites



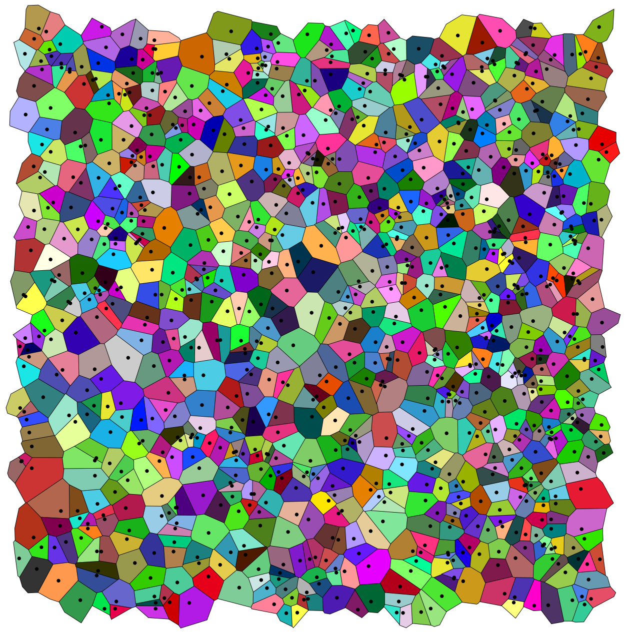
Voronoi cell of site:

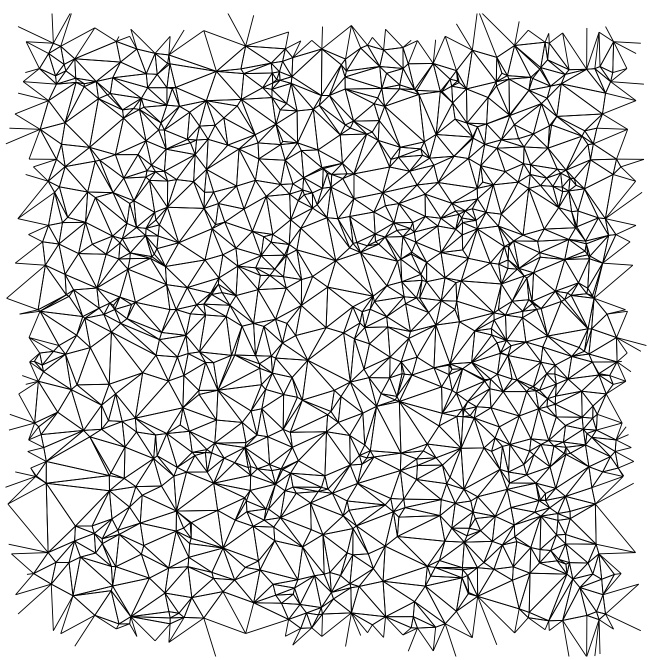
- contains all points in the plane (in space) closer to given site than to any other
- sites whose Voronoi cells share an edge (a face) considered neighbors

Delaunay triangulation (tetrahedrization):

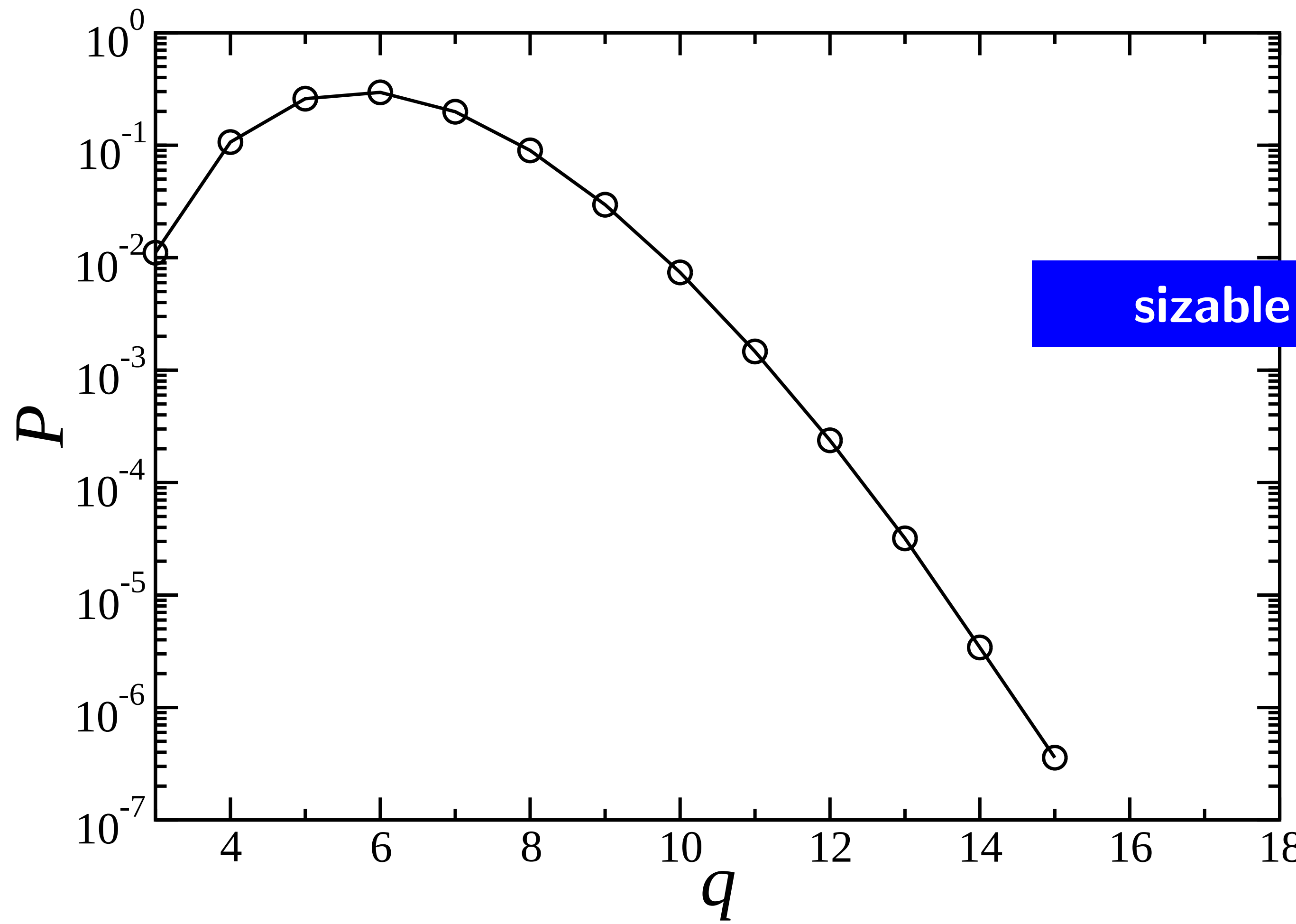
- graph consisting of all bonds connecting pairs of neighbors
- dual lattice to Voronoi lattice

Properties of random Voronoi lattices



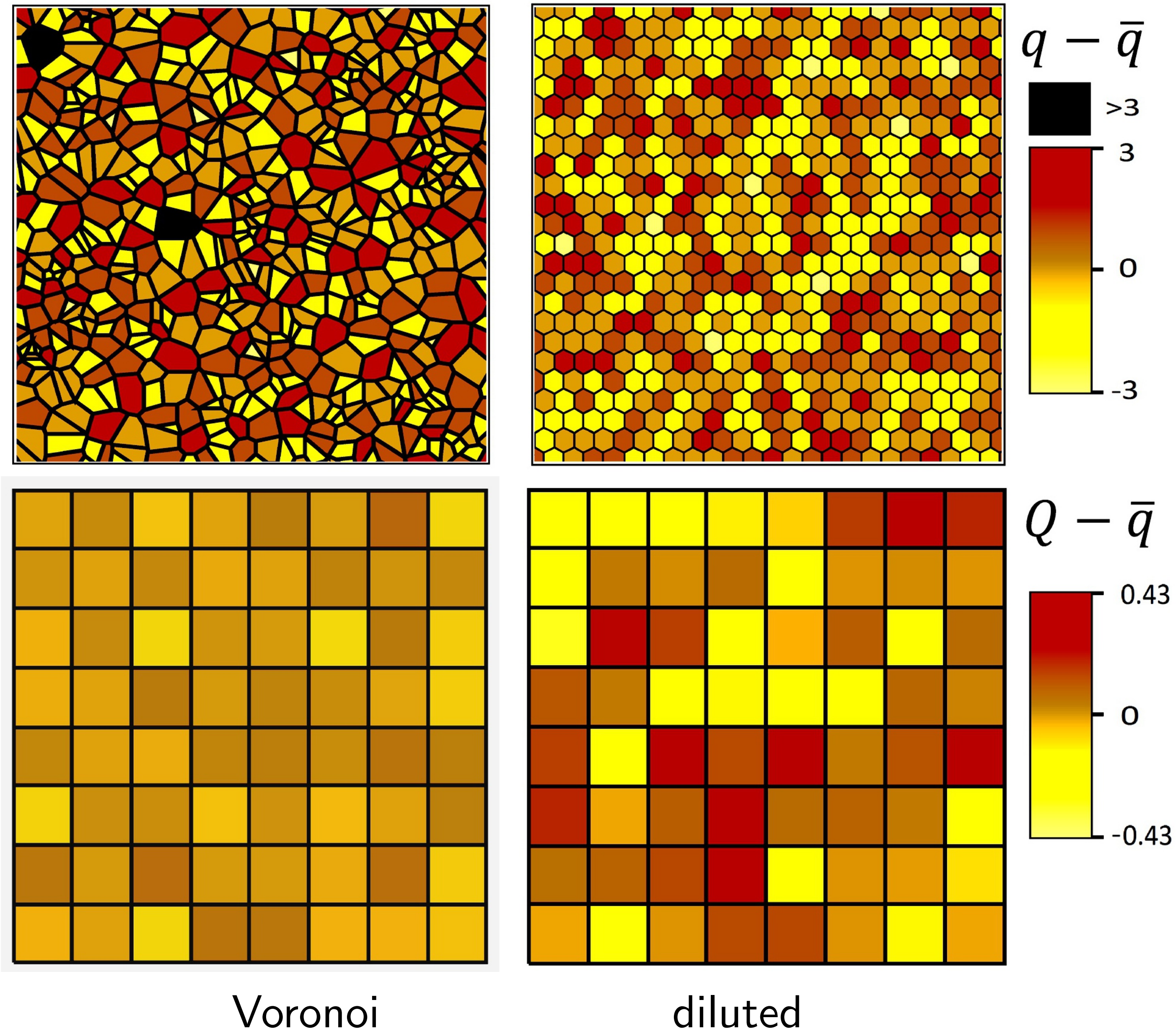


- lattice sites at independent random positions
- local coordination number q_i fluctuates:
  2d: $\langle q \rangle = 6, \qquad \sigma_q \approx 1.33$
  3d: $\langle q \rangle = 2 + (48/35)\pi^2 \approx 15.54, \qquad \sigma_q \approx 3.36$



sizable disorder

Coordination number fluctuations

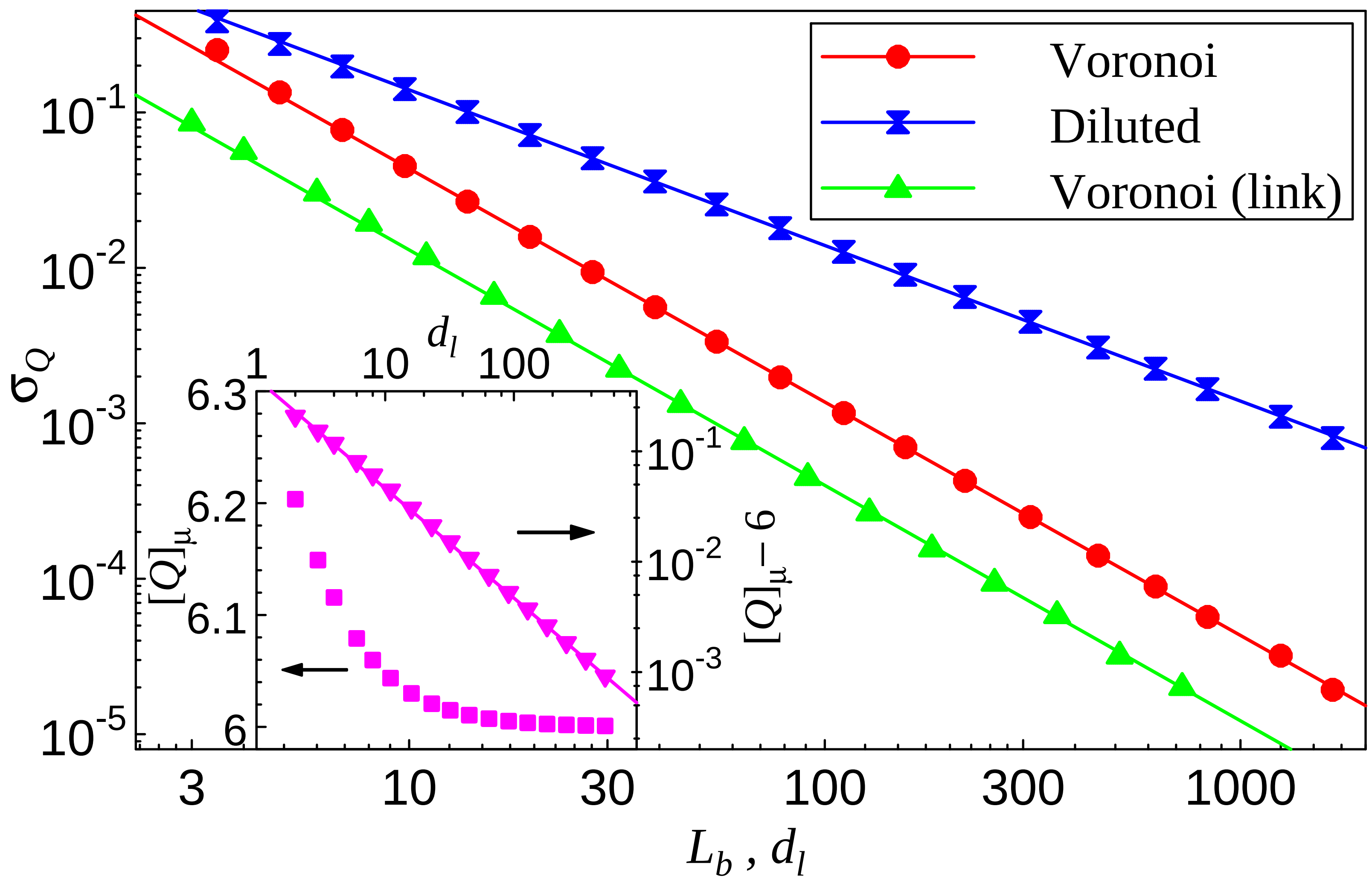


- divide large system into blocks of size L_b
- Calculate block-average coordination number

$$Q_\mu = \frac{1}{N_{b,\mu}} \sum_{i \in \mu} q_i$$

- fluctuations in Voronoi lattice suppressed

Coordination number fluctuations – II



- standard deviation
  $\sigma_Q^2(L_b) = \left[(Q_\mu - \bar{q})^2\right]_\mu$
- **Voronoi lattice:** $\sigma_Q \sim L_b^{-3/2}$
- **diluted lattice:**
  $\sigma_Q \sim L_b^{-1} \sim N_b^{-1/2}$
- also study **link-distance clusters**
- $\sigma_Q \sim L_b^{-3/2}$ as for the real-space clusters

lattice is hyperuniform

Barghathi + Vojta, PRL 113, 120602 (2014)

Topological constraint

- What is the reason for the suppressed disorder fluctuations in the Voronoi lattice??

Euler equation for Delaunay triangulation:
(graph of N lattice sites, E edges, F facets, i.e., triangles)

$$N - E + F = \chi$$

χ: **Euler characteristic**, topological invariant of the underlying surface
torus topology (periodic boundary conditions): $\chi = 0$

- each triangle has three edges, and each edge is shared by two triangles, $3F = 2E$

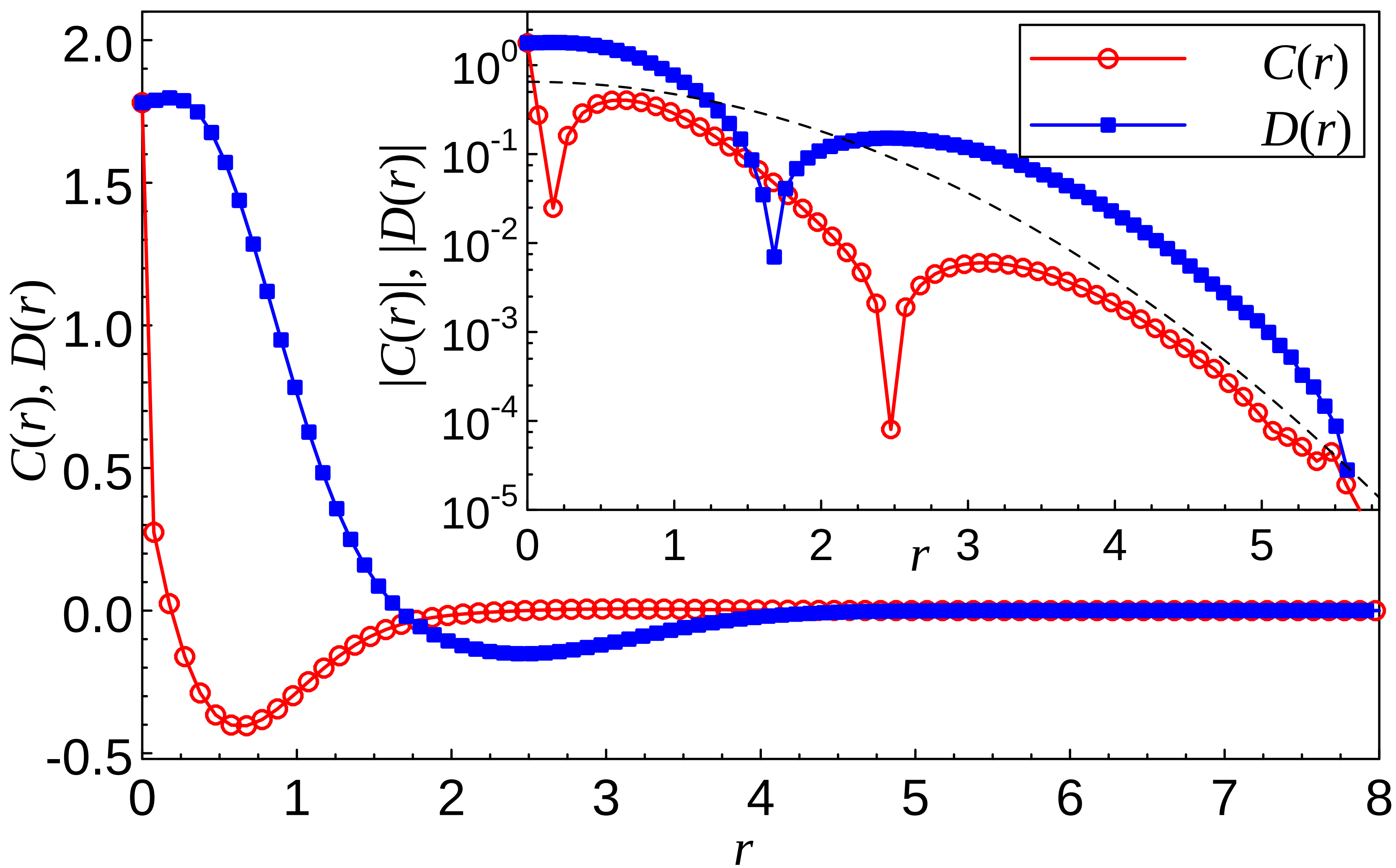
$\Rightarrow$ average coordination number precisely $\bar{q} = 6$ for any disorder configuration

- also follows from fixed angle sum of 180° in a triangle

Topological constraint introduces anticorrelations between disorder fluctuations

- fluctuations stem from surface: $\sigma_Q(L_b) \sim L_b^{(d-1)/2}/L_b^d = L_b^{-(d+1)/2} = L_b^{-3/2}$

Correlation function



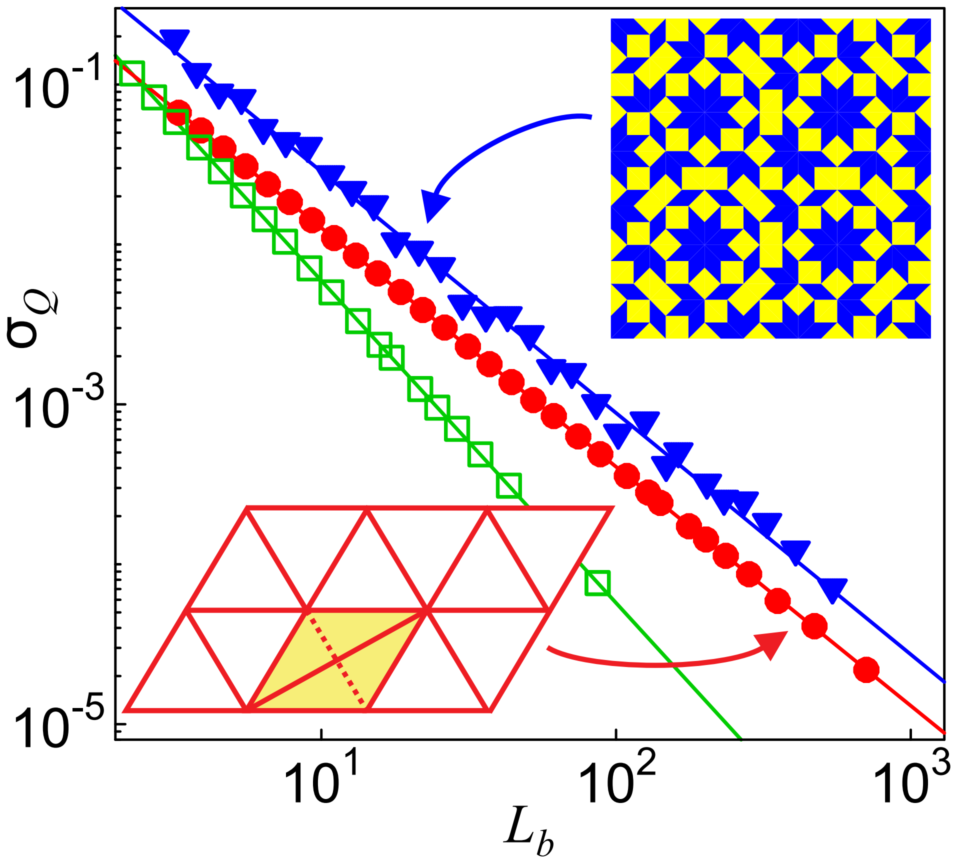
$$C(\mathbf{r}) = \frac{1}{N}\sum_{ij}(q_i - \bar{q})(q_j - \bar{q})\delta(\mathbf{r} - \mathbf{r}_{ij}) \quad , \qquad \sigma^2_{Q,\mathrm{bulk}}(r) = D(r) = \frac{2\pi}{N_r}\int_0^r dr'\, r'\, C(r')$$

$\Rightarrow$ bulk contribution to fluctuations **negligible** beyond 5 or 6 n.n. distances

How general is the suppression of fluctuations?

Two dimensions:

- topological constraint: Euler eq. $N - E + F = \chi$ and triangle relation $3F = 2E$
- holds for all **random triangulations** (with short-range bonds)
- also holds for all **tilings with arbitrary quadrilaterals** (using $4F = 2E$)



Examples:

- random Voronoi lattices
- lattices with random bond-exchange defects
- quasiperiodic Penrose and Ammann-Beenker tilings

broad class of random lattices with fixed total coordination, $\sigma_Q \sim L_b^{-3/2}$

- Random lattices and hyperuniform disorder
- **Superfluid-insulator transition on a random Voronoi lattice**
- Amplitude (Higgs) mode puzzle

Interacting bosons on a random Voronoi lattice

Bose-Hubbard (quantum rotor) Hamiltonian:

$$H = \frac{U}{2} \sum_i (\hat{n}_i - \bar{n}_i)^2 - J \sum_{\langle i,j \rangle} (a_i^\dagger a_j + h.c.)$$

- superfluid ground state if **Josephson coupling** J dominates
- insulating ground state if **charging energy** U dominates

$\Rightarrow$ Superfluid-insulator quantum phase transition as function of U/J

Goals:

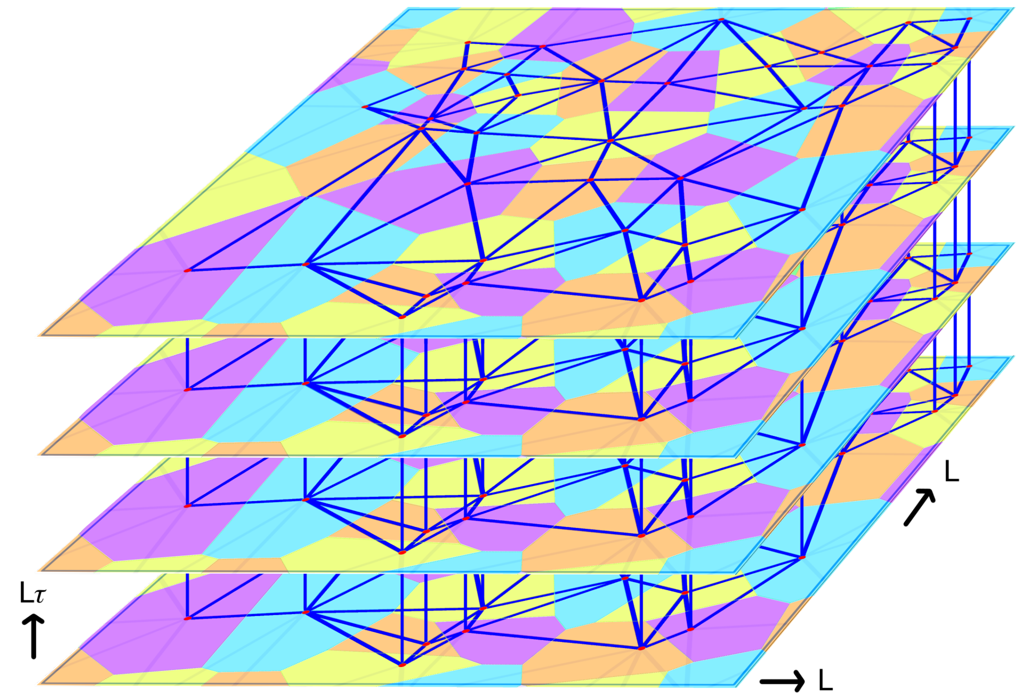
- Understand effects of **hyperuniform disorder** and quantum phase transition
- Solve **Higgs mode puzzle**: What causes the spatial localization of Higgs (amplitude) mode in disordered superfluid

Monte Carlo simulations

- large integer filling $\bar{n}_i$ (**particle-hole symmetric** case): map Hamiltonian onto classical $(2+1)$D XY model

$$H_{\rm cl} = -J_\tau \sum_{i,t} \mathbf{S}_{i,t} \cdot \mathbf{S}_{i,t+1} - J_s \sum_{\langle i,j \rangle, t} \mathbf{S}_{i,t} \cdot \mathbf{S}_{j,t}$$

- combine **Wolff** cluster algorithm and conventional **Metropolis** updates
- large system sizes up to $L = 200$, $L_\tau = 400$
- averages over 1 000 to 5 000 disorder configurations
- **ansiotropic** finite-size scaling analysis



Layered VD lattice, correlated in imaginary time

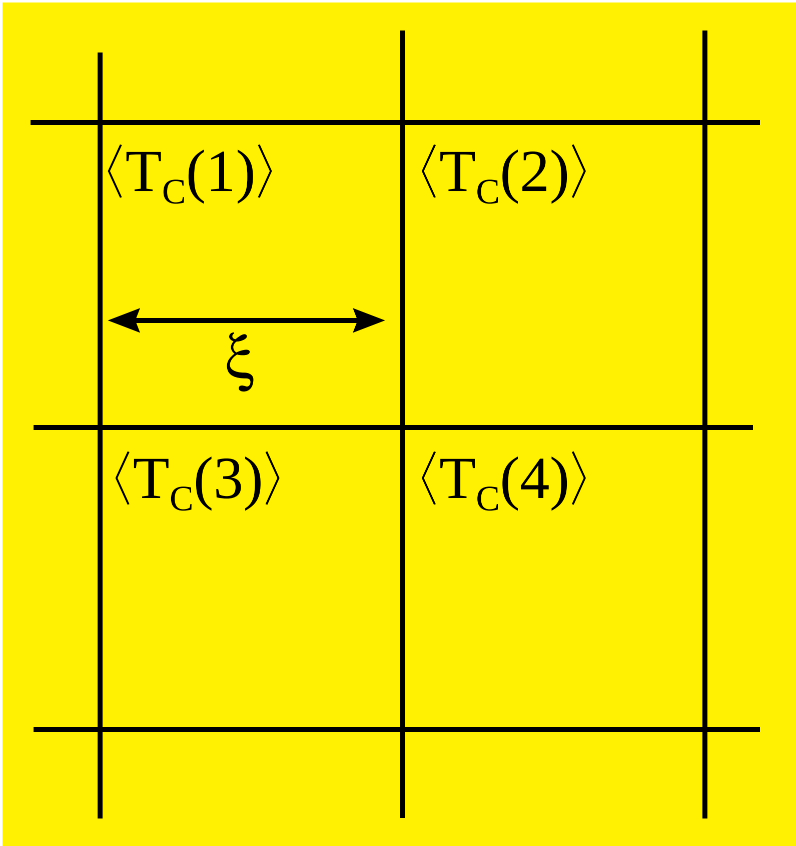
Modified Harris criterion

Stability of clean critical point against randomness:

variation of local $T_c(x)$ between correlation volumes must be smaller than distance from global T_c

Uncorrelated disorder

- variation of average T_c in volume ξ^d: $\Delta\langle T_c(x) :\rangle \sim \xi^{-d/2}$
- distance from global critical point: $|T - T_c| \sim \xi^{-1/\nu}$
- $\Delta\langle T_c(x)\rangle < |T - T_c| \quad \Rightarrow \quad$ stable if $d\nu > 2$



Anticorrelated (hyperuniform) disorder

- variation of average T_c in volume ξ^d: $\Delta\langle T_c(x) :\rangle \sim \xi^{-(d+1)/2}$
- $\Delta\langle T_c(x)\rangle < |T - T_c| \quad \Rightarrow \quad$ stable if $(d+1)\nu > 2$

- clean superconductor-insulator transition features $\nu = 0.6717$

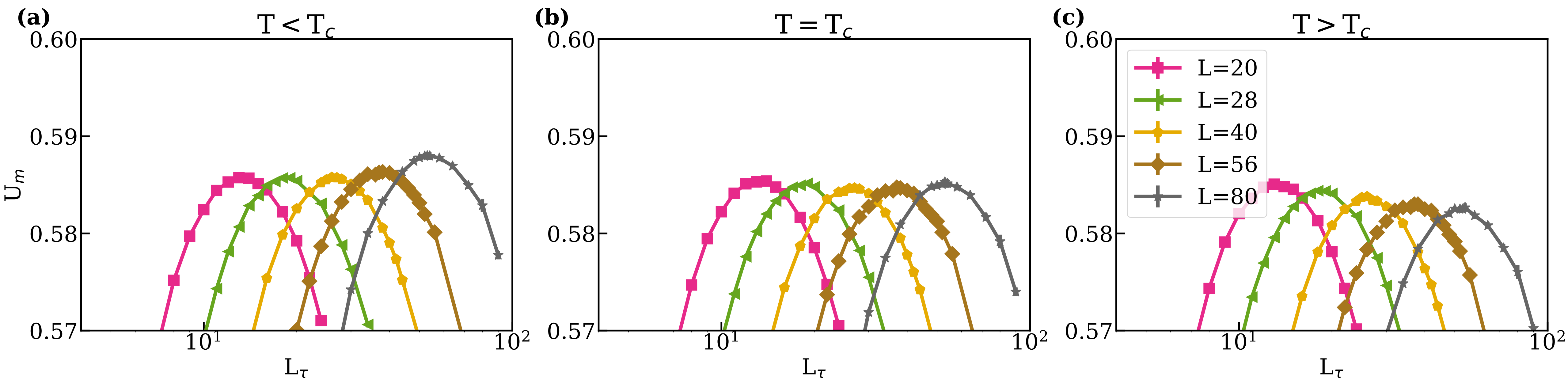
$\Rightarrow$ uncorrelated disorder is **relevant** perturbation but anticorrelated (hyperuniform) disorder is **irrelevant**

Anisotropic finite-size scaling

- disorder breaks **symmetry** between space and (imaginary) time directions
- correct **sample shapes** (aspect ratios between L and L_τ) not known apriori, need to be found during simulation

⇒ **anisotropic finite-size scaling** of the Binder cumulant $U_m = [1 - \langle m^4 \rangle / (3\langle m^2 \rangle^2)]_{\text{dis}}$

- once the "optimal shapes" have been found, finite-size scaling analysis proceeds normally

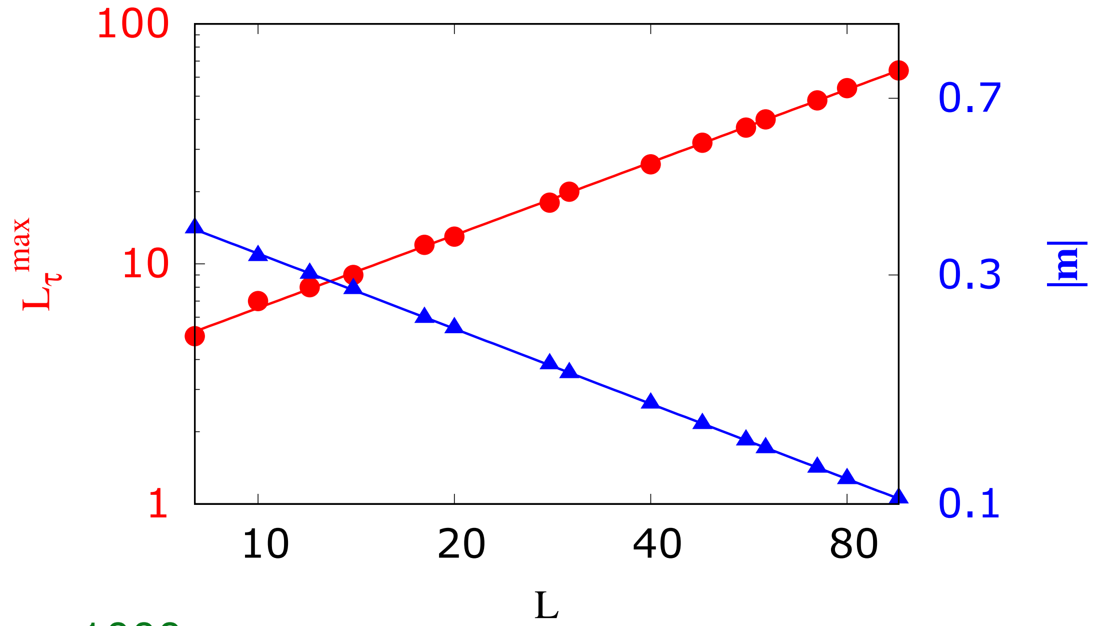


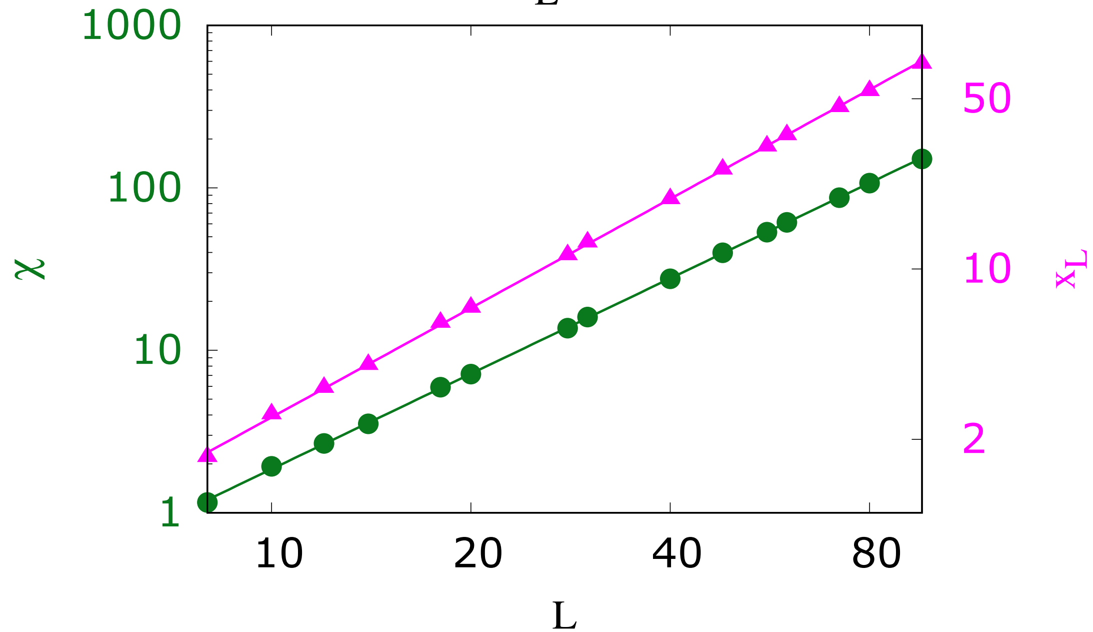
Thermodynamic critical behavior

- Superfluid-insulator transition on random VD lattice features **clean critical behavior**
- agrees with **modified Harris criterion**
- in contrast, generic disorder leads to **new universality class**, in agreement with regular Harris criterion

Exponent	Clean[1]	Generic disorder[2]	VD lattice[3]
ν	0.6717	1.16(5)	0.672(8)
β/ν	0.519	0.48(2)	0.520(4)
γ/ν	1.962	2.52(4)	1.950(10)
z	1	1.52(3)	1.008(9)

[1] Phys. Rev. B 74, 144506 (2006)
[2] Phys. Rev. B 94, 134501 (2016)
[3] Phys. Rev. B 110, 024206 (2024)



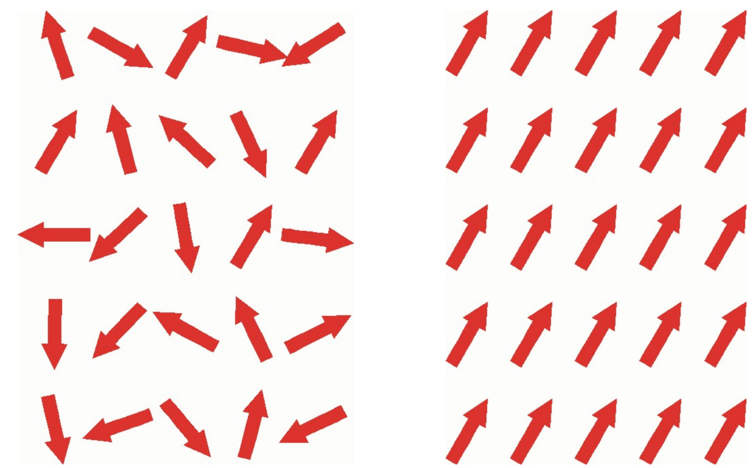


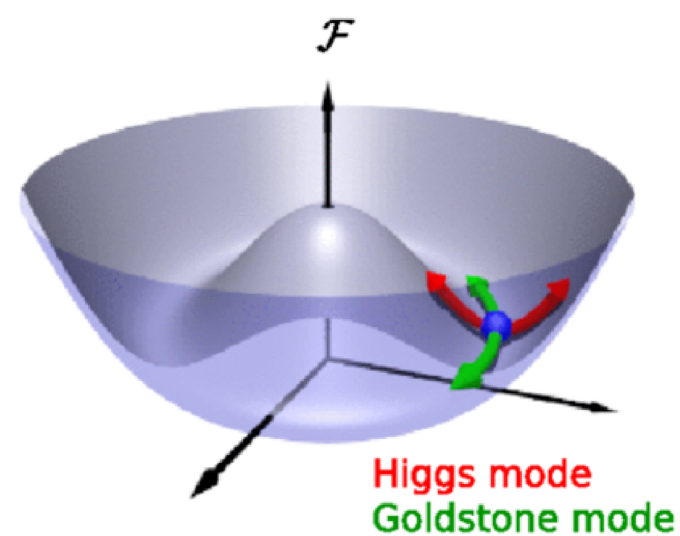
- Random lattices and hyperuniform disorder
- Superfluid-insulator transition on a random Voronoi lattice
- **Amplitude (Higgs) mode puzzle**

Broken symmetries and collective modes

- systems with **broken continuous symmetry**:
  – planar magnet breaks O(2) rotation symmetry
  – superfluid wave function breaks U(1) symmetry
- **Amplitude mode**: corresponds to fluctuations of order parameter **amplitude**
- **Goldstone (phase) mode**: corresponds to fluctuations of order parameter **phase**
- **Amplitude mode** can be considered condensed matter analogue of **Higgs boson**

Goldstone theorem:
When a continuous symmetry is spontaneously broken, massless Goldstone modes appear.





"Mexican hat" potential for order parameter in symmetry-broken phase, $F = t\,\mathbf{m}^2 + u\,\mathbf{m}^4$

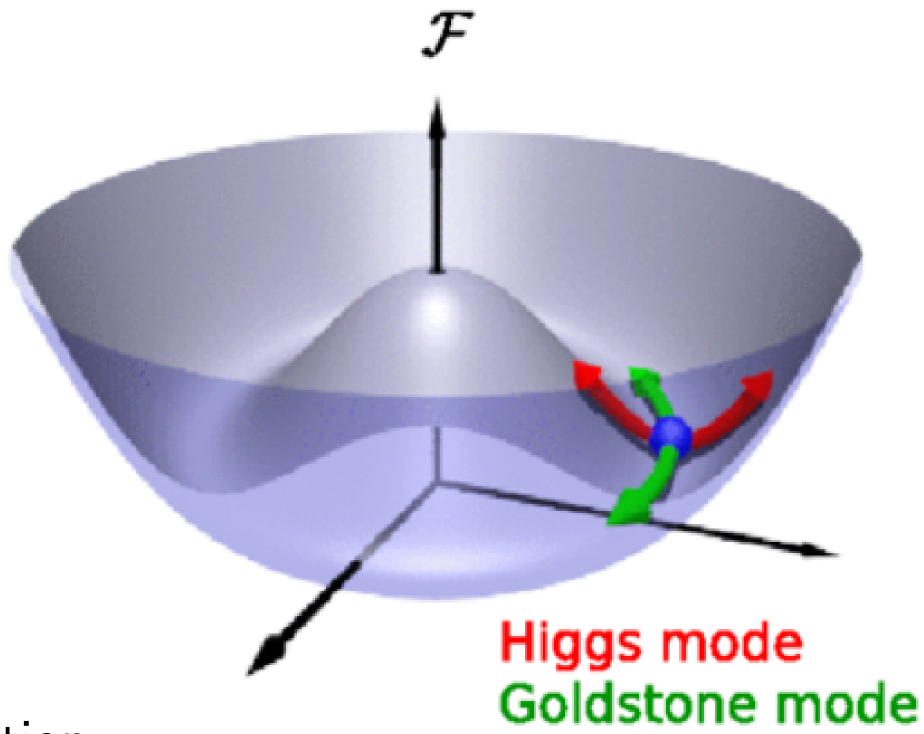
Amplitude mode: scalar susceptibility

- parameterize order parameter fluctuations into **amplitude** and **direction**

$$\vec{\phi} = \phi_0 (1 + \rho) \hat{\mathbf{n}}$$

- Amplitude mode is associated with **scalar** susceptibility

$$\chi_{\rho\rho}(\vec{x}, t) = i\Theta(t) \left\langle [\rho(\vec{x}, t), \rho(0,0)] \right\rangle$$



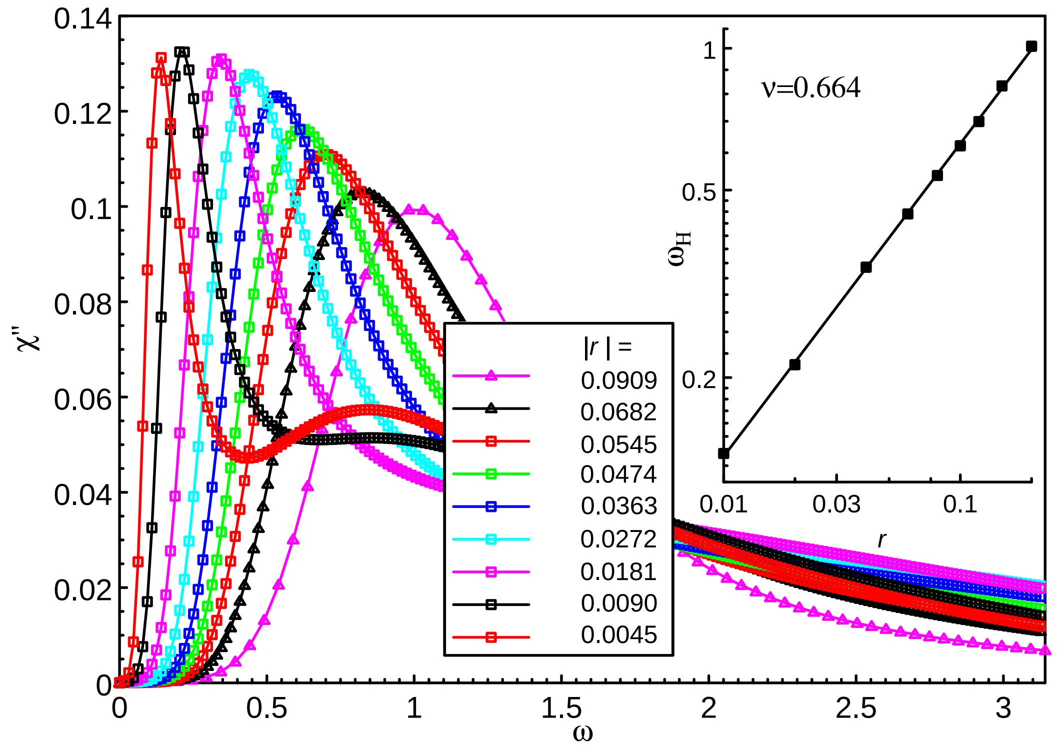
- Monte-Carlo simulations compute **imaginary time** correlation function

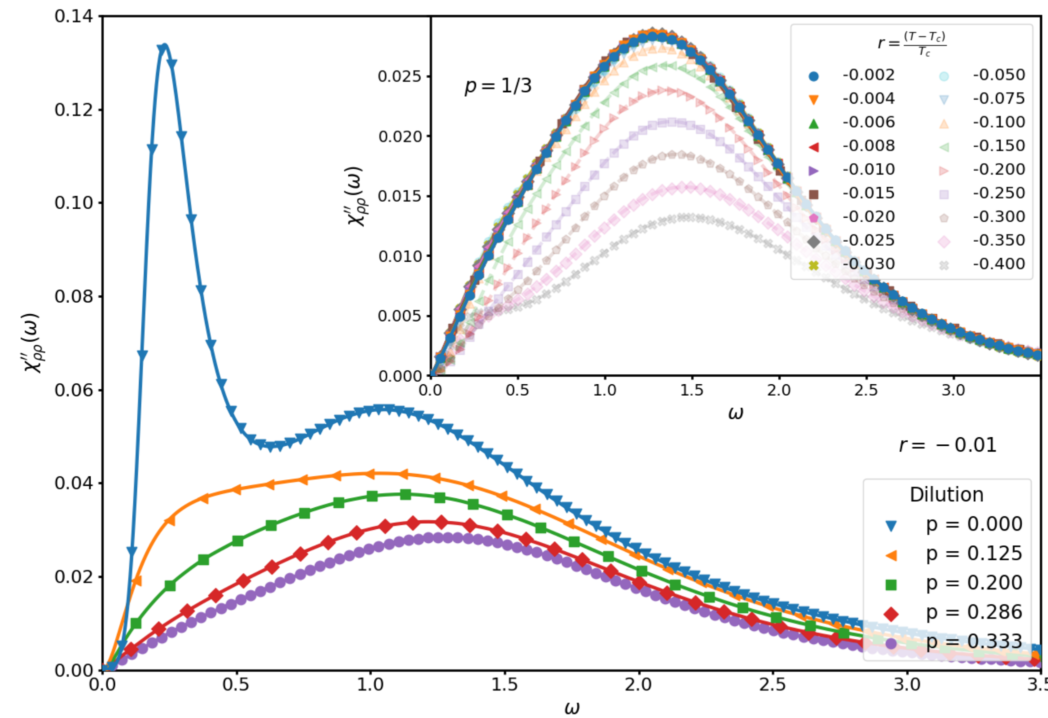
$$\chi_{\rho\rho}(\vec{x}, \tau) = \langle \rho(\vec{x}, \tau) \rho(0,0) \rangle$$

- **Wick rotation** required: analytical continuation from imaginary to real times/frequencies

  $\Rightarrow$ **maximum entropy method** to compute spectral function $A(\vec{q}, \omega) = \chi''_{\rho\rho}(\vec{q}, \omega)/\pi$

Amplitude mode: clean vs. disordered systems





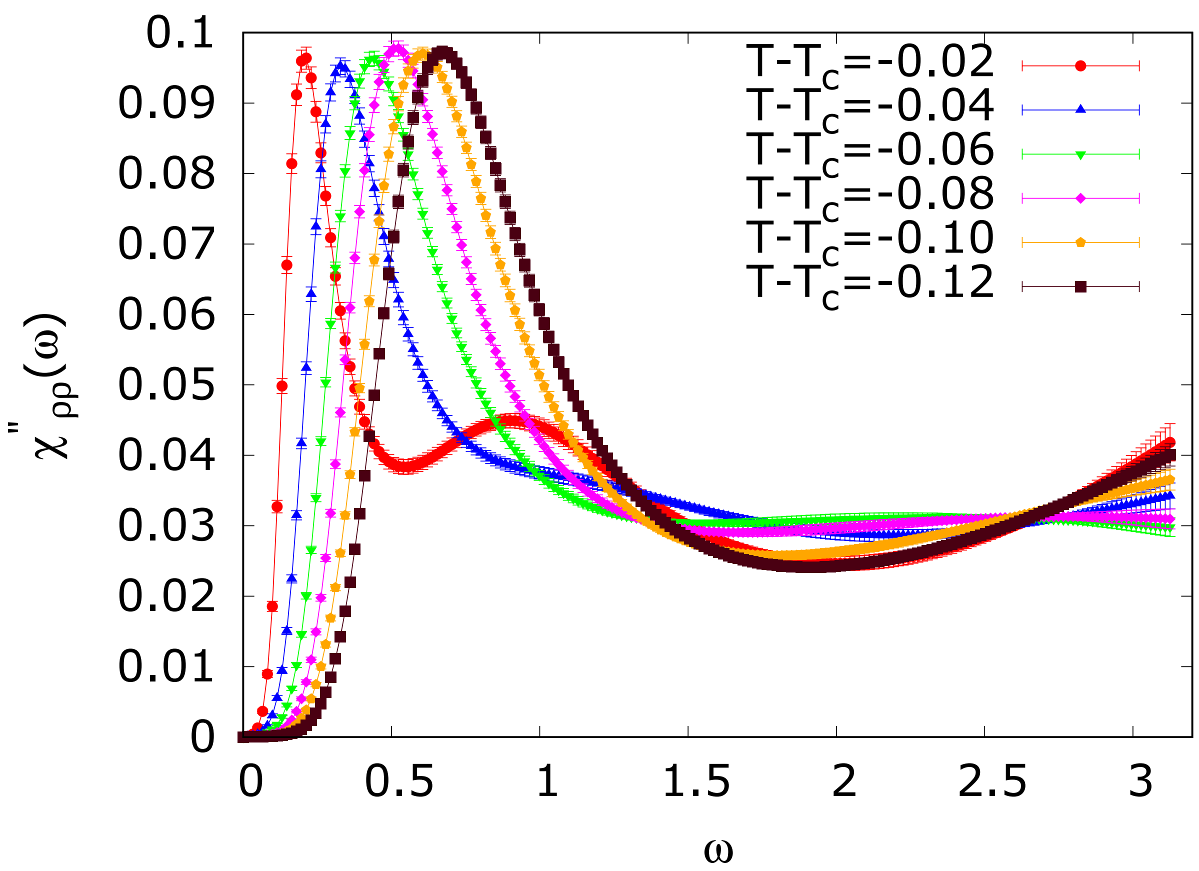
- sharp Higgs peak in spectral function
- **long-lived particle-like excitation**
- **fulfills** scaling form
  $\chi_{\rho\rho}(0,\omega) = |r|^{(d+z)\nu-2} X(\omega|r|^{-z\nu})$

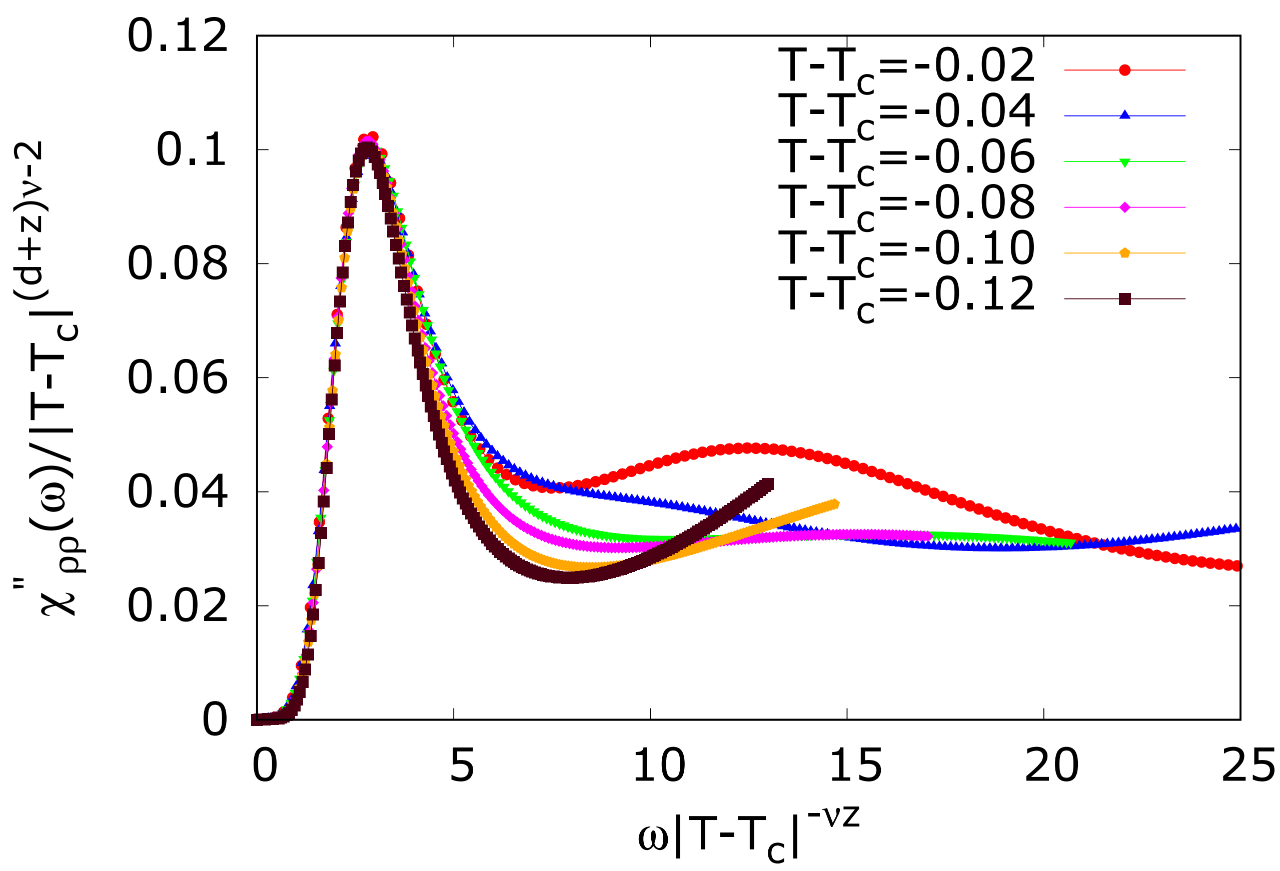
- disorder suppresses sharp Higgs peak
- $\chi_{\rho\rho}$ **violates** naive scaling
- flat energy-momentum dispersion

$\Rightarrow$ amplitude mode **spatially localized**

What causes broad spectral response? Anderson localization or mode-mode interactions

Amplitude mode on a random Voronoi-Delaunay lattice





- **sharp Higgs peaks** as in the clean case
- **fulfills** expected scaling form $\chi_{\rho\rho}(0,\omega) = |r|^{(d+z)\nu-2} X(\omega|r|^{-z\nu})$

⇒ evidence against Anderson localization (non-interacting particles on random VD lattice are fully localized)

⇒ character of amplitude mode governed by critical behavior of the transition (**scale dimension** of $\chi_{\rho\rho}$)

Conclusions

- broad class of random lattices are **hyperuniform** with strong disorder **anticorrelations**, including random Voronoi-Delaunay lattice
- critical points on such lattices are governed by **modified Harris criterion** $(d+1)\nu > 2$
- superfluid-insulator transition on random Voronoi-Delaunay lattice shows **clean** critical behavior
- amplitude (Higgs) mode remains **sharp, delocalized**, particle-like excitation
- amplitude localization in the presence of conventional disorder not driven by Anderson localization but by **mode-mode interaction** effects, governed by the critical fixed point

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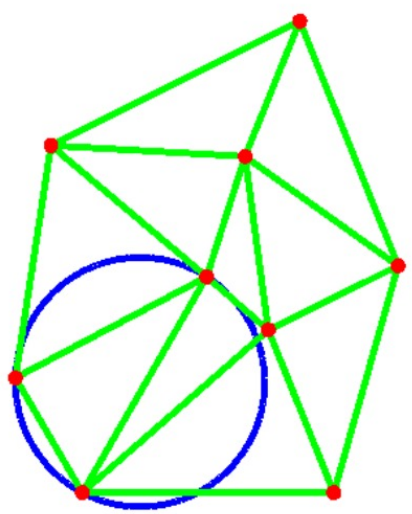
Superfluid transition on VD lattice: Phys. Rev. B **110**, 024206 (2024)

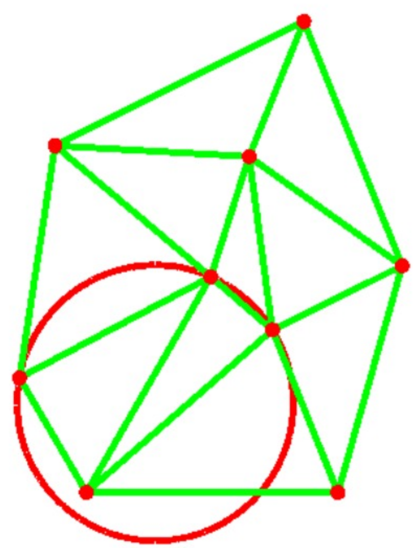
Amplitude mode localization: Phys. Rev. Lett. **125**, 027002 (2020), Phys. Rev. B **104**, 014511 (2021)

Random lattices and modified Harris criterion: Phys. Rev. Lett. **113**, 120602 (2014)

Algorithms

- generating Voronoi lattice or Delaunay triangulation is prototypical problem in **computational geometry**
- many different algorithms exist
- efficient algorithm inspired by Tanemura et al., uses **empty circumcircle property**
  up to 5000^2 sites in 2d and 400^3 sites in 3d
- computer time scales roughly **linearly** with number of sites
- 10^6 sites in 2d: about 30 seconds on PC
  10^6 sites in 3d: about 3 min

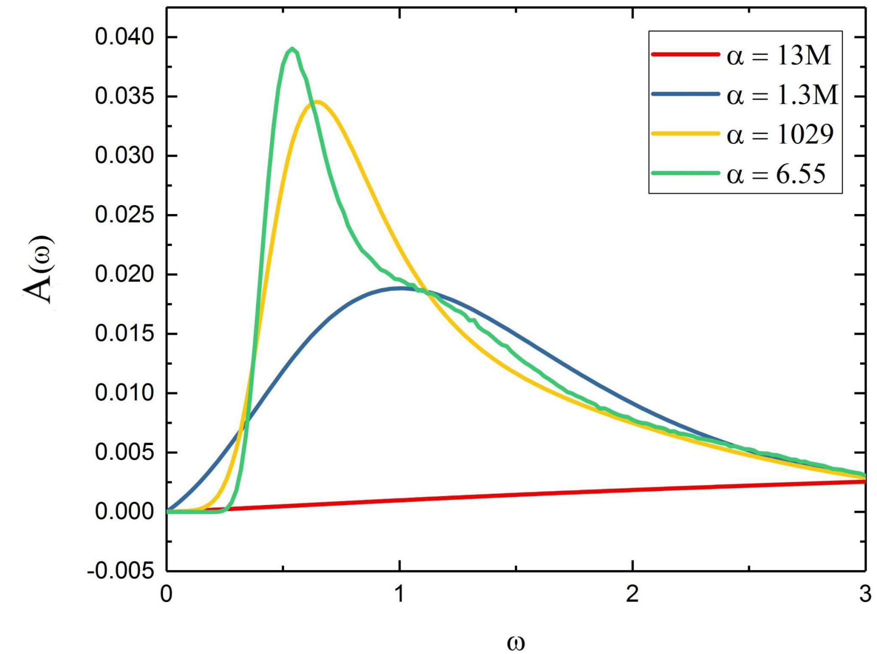




Analytic continuation - maximum entropy method

- **Matsubara susceptibility** vs. **spectral function**

$$\chi_{\rho\rho}(\vec{q}, i\omega_m) = \int_0^\infty d\omega A(\vec{q}, \omega) \frac{2\omega}{\omega_m^2 + \omega^2}$$



Maximum entropy method:

- inversion is ill-posed problem, highly sensitive to noise
- fit $A(\vec{q}, \omega)$ to $\chi_{\rho\rho}(\vec{q}, i\omega_m)$ MC data by minimizing $Q = \frac{1}{2}\sigma^2 - \alpha S$
- parameter α balances between fit error σ^2 and entropy S of $A(\vec{q}, \omega)$, i.e., between fitting information and noise
- best α value chosen by L-curve method [see Bergeron et al., PRE 94, 023303 (2016)]